PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Scattering of a Scalar Wave by a Semi-Infinite Rod of Circular Cross Section

D. S. Jones

Phil. Trans. R. Soc. Lond. A 1955 **247**, 499-528
doi: 10.1098/rsta.1955.0004

THE SCATTERING OF A SCALAR WAVE BY A SEMI-INFINITE ROD OF CIRCULAR CROSS SECTION

By D. S. JONES

Department of Mathematics, University of Manchester

(*Communicated by M. J. Lighthill, F.R.S.—Received* 9 *June* 1954)

CONTENTS

The form of the exact solution for the scattering of a plane harmonic scalar wave by a semi-infinite circular cylindrical rod of diameter $2a$ is found when the boundary condition is $u=0$ or $\partial u/\partial \nu = 0$, where u represents the scalar field and ν is the normal to the rod.

When the angle of incidence is π, i.e. the angle between the direction of propagation of the incident wave and the normal (out of the rod) to the end is π, the average pressure amplitude on the end of the rod and the scattering coefficient are found for the boundary condition $\partial u/\partial \nu = 0$. Graphs are given showing the behaviour of these quantities for the range $0 \leqslant ka \leqslant 10$, where k is the wave-number. When ka reaches 10, the quantities have almost become constant. For small values of ka the scattering coefficient is shown to be $\frac{1}{4}(ka)^2$; it appears from the numerical results that this is, in fact, a fairly close approximation for $ka < 2$.

It is further shown that the average pressure amplitude on the end for other angles of incidence is approximately the product of the average pressure amplitude for an angle of incidence of π and the amplitude of the symmetric mode ($ka < 3{\cdot}83$) which the incident field would produce inside a hollow semi-infinite cylinder occupying the same position as the rod.

When the boundary condition is $u=0$ and ka is small it is proved that the scattered field is the same as that due to a semi-infinite hollow cylinder longer by an amount $0{\cdot}1a$ approximately. A similar result does not hold for the boundary condition $\partial u/\partial \nu = 0$.

The theory is extended to the case when a pressure pulse falls on a circular rod. It is found that the pressure on the end drops almost to its final value in the time taken for a wave to travel the diameter of the rod, and that the average pressure during this process is given, at time t, by $\{0{\cdot}915 + 0{\cdot}745(2 - a_0 t/a)^2\}^{\frac{1}{2}}$ approximately, where a_0 is the speed of sound.

Tables, in the range $0(0{\cdot}25)10$ of ka, of the 'split' functions which arise in connexion with a semi-infinite cylinder are given.

Introduction

An exact solution of the problem of the scattering by a dielectric cylindrical rod of finite length is not known, although this problem is of importance in light-scattering experiments, the design of antennae and the design of microphones. When the length of the rod is infinite the solution is well known, but our theoretical knowledge of the scattering due to finite obstacles in three dimensions is essentially limited to those of ellipsoidal shape. (For two-dimensional obstacles information has recently been published (Jones 1953 *a*) concerning the scattering by impenetrable rectangles.) However, it is possible to make some progress by the introduction of various simplifications. The first simplification is to make the rod semi-infinite in length. This restriction is not so severe as appears at first sight, since it is frequently possible to predict the behaviour of finite obstacles from that of semi-infinite obstacles (see, for example, Jones 1952 *a* and Williams 1954 *a*).

The second simplification is to assume that the fields cannot penetrate the rod, i.e. the boundary condition on the rod is either $u = 0$ or $\partial u/\partial \nu = 0$; this is often physically reasonable. More account can be taken of the dielectric nature of the rod by using the boundary condition $\partial u/\partial \nu + \sigma u = 0$; the author hopes to consider this in the near future.

The first boundary condition which is considered is $\partial u/\partial \nu = 0$; this corresponds to the problem of small-amplitude sound waves falling on a rigid rod. The problem when the angle of incidence is π is examined in some detail numerically, the angle of incidence being the angle between the direction of propagation of the incident wave and the normal (out of the rod) to the end. The average pressure amplitude on the end is found to be related to one of a set of constants which satisfy an infinite number of equations. Three approximations have been made in which it is assumed that only one, two or three of the constants are non-zero. The results of the second and third approximations scarcely differ, so that it may be assumed that they give a close estimate of the average pressure amplitude but no analytical justification is attempted. A variational method of still further improving the approximation is given but no advantage of this has been taken in the numerical work.

The second and third approximations show that the average pressure amplitude rises steadily, as ka increases, from the value 1 (the pressure amplitude in the incident wave being 1) when $ka = 0$ to a maximum of 2·18 when $ka = 2{\cdot}4$ and then oscillates with decreasing amplitude about the value 2.

The same three approximations have also been used to determine the scattering coefficient. For small values of ka the scattering coefficient is effectively $\frac{1}{4}(ka)^2$. It increases steadily with ka (very nearly like $\frac{1}{4}(ka)^2$) up to a maximum of 1·14 at $ka = 2{\cdot}5$ and thereafter oscillates slightly about the value 1. For $ka > 3$ the scattering coefficient is almost half the scattering coefficient for a rigid disk of diameter $2a$. The difference of a half is accounted for by the fact that the field can be diffracted round behind the disk, but this is not possible for the rod.

When the angle of incidence is not 0 or π it is shown that, to a first approximation (which is better the smaller ka), the average pressure amplitude on the end of the cylinder is the product of the average pressure amplitude when the angle of incidence is π and the amplitude of the symmetric wave that is produced in a hollow semi-infinite cylinder occupying the same position as the rod. The solution for the hollow semi-infinite cylinder has been given by Levine & Schwinger (1948 *a*). This result has been used to draw graphs of the average pressure amplitude for various angles of incidence in the range $0 \leqslant ka \leqslant 3{\cdot}5$. The approximation would not be expected to hold outside this range and, in any case, the pressure can be estimated on the basis of geometrical optics for $ka \geqslant 4$ when the incident wave 'illuminates' the end of the rod. As the angle of incidence decreases from π to $\frac{1}{2}\pi$ the first maximum of the average pressure amplitude decreases and occurs nearer to $ka = 0$, being

1·1 at $ka = 1{\cdot}25$ for an angle of incidence of $\frac{2}{3}\pi$. For angles of incidence less than $\frac{1}{2}\pi$ the average pressure amplitude drops steadily from its value of 1 at $ka = 0$ as ka increases. One curious feature, for which the author has no explanation, is that the average pressure amplitude for an angle of $\frac{1}{6}\pi$ is greater than that for $\frac{1}{3}\pi$ for all ka under consideration.

The second boundary condition which has been considered is $u = 0$. This has not been examined in such detail numerically as the preceding boundary condition, not because it is less important but because there was a limit to the computation the author was prepared to undertake. The boundary condition $u = 0$ is appropriate to the antenna problem in so far as it is permissible to regard the component of the electric intensity parallel to the axis of the rod as the only important component. It is shown that the cylindrical rod of small diameter behaves as a cylindrical tube of the same diameter but longer by an amount $0{\cdot}1a$ as regards the distant field that is produced. This means also that the current along the rod at some distance from the end is the same as that of the longer tube. (No wave propagates inside the tube which is beyond cut-off.) It follows that a non-resonant rod of finite length of small diameter may be treated, as regards the current at its centre and the radiated field, as a tube of slightly longer length. Such an approximation should be better than the standard method of replacing the rod by a line source and applying the boundary condition on the surface of the rod. The solution for the finite tube may be obtained in a similar way to that used by Williams (1954*b*) for sound waves.

It should be noted that, although the analysis estimating the extra length is only strictly valid for diameters less than $\frac{1}{10}$ wave-length, the theory and experiment for a thick plate did not differ by more than a factor of 2 for a thickness of a half wave-length, and so it is possible that like results may well hold in this case. (The agreement is not likely to hold for such a wide range in the particular application to the antenna problem, because the effect of other components of the electric intensity becomes more significant as the diameter of the end increases.)

There is no comparable result that the small-diameter rod is equivalent to the longer tube when the boundary condition is $\partial u/\partial \nu = 0$. This is because the normal derivative of the first-order cylindrical harmonic of the incident field is as large on the cylindrical surface as that of the zero order harmonic when ka is small; the effective lengthening for the two harmonics is different.

It is possible to deduce, from the analysis required for the above problems, the average pressure due to a pressure pulse falling on a rigid rod. It is found that the main pressure drop takes place while the waves first diffracted at the edges are spreading across the end of the rod. During this time the average pressure on the end is $\{0{\cdot}915+0{\cdot}745(2-a_0t/a)^2\}^{\frac{1}{2}}$ when the pressure in the incident wave is unity. The result also holds for finite cylinders whose length is greater than the radius and also for a semicircular obstacle on a rigid plane. This last problem is related to that of finding the force on a Nissen hut due to a weak shock wave travelling along the earth.

The technique used in solving the problem of the semi-infinite rod is the same as that which the author has already used in connexion with the diffraction by a thick plate (Jones 1953*a*). We first obtain the solution for the boundary condition $\partial u/\partial \nu = 0$ when the angle of incidence is not 0 or π and deduce from it the solution when the angle of incidence is π. The next section is concerned with the boundary condition $u = 0$.

In § 4 are obtained expressions for the distant field, and, in § 5, formulae for the average pressure amplitude and scattering coefficient are found. § 6 is devoted to obtaining variational expressions for some of the quantities to be computed.

The behaviour of the distant field when $ka \ll 1$ and the boundary condition is $u = 0$ is considered in § 7, the results being interpreted in the manner already described. The behaviour of a pressure pulse is discussed in § 8.

The last section carries an account of some approximations which were used in computation and also occasionally in the text. An appendix contains properties of various functions occurring and references to it are indicated by (A 1), for example, meaning equation (1) of the appendix.

1. The boundary condition $\partial u/\partial \nu = 0$ when the angle of incidence is not 0 or π

Let ρ, ϕ, z be cylindrical polar co-ordinates and let the semi-infinite rod occupy the space $\rho \leqslant a$, $z \leqslant 0$. Let the incident plane wave $u^{(0)}$ be given by

$$u^{(0)} = \exp\left(-\mathrm{i}k\rho\sin\theta\cos\phi - \mathrm{i}kz\cos\theta\right),$$

where $0 < \theta < \pi$ and $k = 2\pi/(\text{wave-length})$. The field that is produced by this plane wave falling on the rod must satisfy

$$\nabla^2 u + k^2 u = 0, \tag{1}$$

where

$$\nabla^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2},$$

and be such that $\partial u/\partial\nu = 0$ on the surface of the rod.

The expression for $u^{(0)}$ may be rewritten

$$u^{(0)} = \mathrm{e}^{-\mathrm{i}kz\cos\theta} \sum_{n=-\infty}^{\infty} u_n^{(0)}(\rho)\,\mathrm{e}^{\mathrm{i}n\phi}, \tag{2}$$

where $u_n^{(0)} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}n\pi} J_n(k\rho\sin\theta)$, J_n being the Bessel function of the first kind and nth order. The field $u^{(1)}$ which would be reflected if the cylinder occupied the whole of $\rho \leqslant a$, $-\infty < z < \infty$ is given by

$$u^{(1)} = \mathrm{e}^{-\mathrm{i}kz\cos\theta} \sum_{n=-\infty}^{\infty} u_n^{(1)}(\rho)\,\mathrm{e}^{\mathrm{i}n\phi}, \tag{3}$$

where

$$u_n^{(1)}(\rho) = \mathrm{e}^{-\frac{1}{2}\mathrm{i}n\pi}\frac{J_n'(ka\sin\theta)}{H_n^{(2)\prime}(ka\sin\theta)} H_n^{(2)}(k\rho\sin\theta),$$

where $H_n^{(2)}$ is the Hankel function of the second kind and nth order and primes indicate derivatives with respect to the argument. It is clear that

$$\partial u^{(0)}/\partial\rho = \partial u^{(1)}/\partial\rho \tag{4}$$

on $\rho = a$ for all z.

Let the total field be given by

$$\begin{array}{lll} u^{(0)} - u^{(1)} + u(\rho,\phi,z) & \text{in} & \rho > a, \\ u(\rho,\phi,z) & \text{in} & \rho < a,\ z > 0, \\ 0 & \text{in} & \rho < a,\ z < 0. \end{array}$$

Both $u^{(0)}$ and $u^{(1)}$ satisfy (1) and hence u satisfies (1). The other conditions to be satisfied by u are

(i) $\partial u/\partial \nu = 0$ on the surface of the rod on account of (4) and the assumed form of the total field;

(ii) $\partial u/\partial \rho$ is continuous across $\rho = a$ for all z except possibly $z = 0$;

(iii) the total field is continuous except possibly across the surface of the rod;

(iv) $u \sim \text{const. } \mathrm{e}^{-\mathrm{i}kr}/r$ as $r = (\rho^2+z^2)^{\frac{1}{2}} \to \infty$ in $\rho > a$ and $u \sim \text{const. } \mathrm{e}^{-\mathrm{i}kz\cos\theta}$ as $z \to \infty$ in $\rho < a$;

(v) $u = \text{const.} + O(\varpi^{\frac{2}{3}})$, $|\operatorname{grad} u| = O(\varpi^{-\frac{1}{3}})$ as $\varpi \to 0$, where ϖ is the distance between a point of $\rho = a$, $z = 0$ and a point of observation with the same ϕ.

These conditions are sufficient to ensure a unique solution (Jones 1952*b*; Meixner 1949).

We assume, for analytical convenience, that $k = k_r - \mathrm{i}k_i$ ($k_r > 0$, $k_i > 0$) and then allow $k_i \to 0$ when the analysis is complete. That such a process leads to a solution of our problem may be verified *a posteriori*. In the corresponding problem when the boundary condition is $u = 0$ this verification is not necessary because, in this case, the spectrum of $-\nabla^2$ is continuous (see theorem 9 of Jones 1953*b*).

Let s be the complex variable $\sigma + \mathrm{i}\tau$ and define $U(\rho, \phi, s)$ by

$$U(\rho,\phi,s) = \int_{-\infty}^{\infty} u(\rho,\phi,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z.$$

It follows from (iv) that U is analytic in the strip $-k_i\cos\theta < \sigma < k_i$.

Define u_n and U_n by

$$u_n(\rho,z) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\phi,z)\,\mathrm{e}^{-\mathrm{i}n\phi}\,\mathrm{d}\phi, \quad U_n(\rho,s) = \frac{1}{2\pi}\int_0^{2\pi} U(\rho,\phi,s)\,\mathrm{e}^{-\mathrm{i}n\phi}\,\mathrm{d}\phi.$$

Then, since u satisfies (1), it follows from (i) that

$$\left.\begin{aligned}\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}U_n}{\mathrm{d}\rho}\right) + \left(\kappa^2 - \frac{n^2}{\rho^2}\right)U_n &= 0 \qquad (\rho > a),\\ &= sf_n(\rho) \quad (\rho < a),\end{aligned}\right\} \tag{5}$$

where $f_n(\rho) = \lim\limits_{z\to+0} u_n$ and $\kappa^2 = s^2 + k^2$. Equations (5) hold in the strip $-k_i\cos\theta < \sigma < k_i$. We specify κ by choosing that branch of $(s^2+k^2)^{\frac{1}{2}}$ which reduces to k when $s = 0$. Then κ has a negative imaginary part in the strip $-k_i < \sigma < k_i$.

The solution of (5) in $\rho > a$ which complies with (iv) is

$$U_n(\rho,s) = Q_n(s)\,H_n^{(2)}(\kappa\rho)/\kappa, \tag{6}$$

where $Q_n(s)$ is independent of ρ.

In $\rho < a$ choose the particular integral which is bounded and whose derivative vanishes on $\rho = a$. Then the bounded solution of (5) in $\rho < a$ is

$$U_n(\rho,s) = \{1/J_n'(\kappa a)\}\{P_n(s)\,J_n(\kappa\rho)/\kappa + s\int_0^a tf_n(t)\,p_n(t,\rho)\,\mathrm{d}t\}, \tag{7}$$

where

$$\begin{aligned}p_n(t,\rho) &= \tfrac{1}{2}\pi\{J_n'(\kappa a)\,Y_n(\kappa\rho) - Y_n'(\kappa a)\,J_n(\kappa\rho)\}J_n(\kappa t) \quad (0 \leqslant t \leqslant \rho),\\ &= \tfrac{1}{2}\pi\{J_n'(\kappa a)\,Y_n(\kappa t) - Y_n'(\kappa a)\,J_n(\kappa t)\}J_n(\kappa\rho) \quad (\rho \leqslant t \leqslant a),\end{aligned}$$

Y_n being the Bessel function of the second kind and nth order.

The combination of (ii) and (v) shows that $\frac{\partial}{\partial \rho} U(\rho, \phi, s)$ is continuous across $\rho = a$ and hence $\frac{\mathrm{d}}{\mathrm{d}\rho} U_n(\rho, s)$ is continuous across $\rho = a$. Hence we have, from (6) and (7),

$$Q_n(s)\, H_n^{(2)\prime}(\kappa a) = P_n(s).$$

Therefore

$$U_n(\rho, s) = P_n(s)\, H_n^{(2)}(\kappa\rho)/\kappa H_n^{(2)\prime}(\kappa a) \tag{8}$$

in $\rho > a$ and $\frac{\mathrm{d}}{\mathrm{d}\rho} U_n(a, s) = P_n(s)$. But $\partial u/\partial\rho = 0$ on $\rho = a$, $z < 0$ by (i) and so

$$P_n(s) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\phi}\,\mathrm{d}\phi \int_0^\infty \frac{\partial}{\partial\rho} u(a, \phi, z)\, \mathrm{e}^{-sz}\,\mathrm{d}z.$$

Hence $P_n(s)$ is analytic in $\sigma > -k_i \cos\theta$. If the domains $\sigma > -k_i\cos\theta$, $\sigma < k_i$ are designated the positive and negative half-planes respectively it can be said that $P_n(s)$ is analytic in the positive half-plane.

In $\rho < a$, u vanishes for $z < 0$, and hence $U_n(\rho, s)$ must be analytic in the positive half-plane. Therefore the right-hand side of (7) must not have poles at the zeros of $J_n'(\kappa a)$ which lie in the positive half-plane.

Let

$$J_n'(j_{nm}') = 0 \quad \text{where} \quad j_{nm}' > j_{n,m-1}' \quad \text{and} \quad j_{n0}' = 0.$$

Write

$$\kappa_{nm}^2 = j_{nm}'^2/a^2 - k^2,$$

and define κ_{nm}, when $k_i = 0$, to be positive when real and positive imaginary otherwise. It then follows that

$$\mathscr{R}(\kappa_{nm}) \geqslant k_i, \quad \mathscr{I}(\kappa_{nm}) \leqslant k_r,$$

the equality holding only when $m = 0$. Hence the zeros of $J_n'(\kappa a)$ in the positive half-plane are $s = \kappa_{nm}$ $(m = 0, 1, \ldots)$.

In order that $U_n(\rho, s)$ shall be analytic in the positive half-plane it is necessary that

$$\frac{a}{j_{nm}'} P_n(\kappa_{nm})\, J_n\!\left(\frac{j_{nm}'\rho}{a}\right) - \tfrac{1}{2}\pi\kappa_{nm}\, Y_n'(j_{nm}')\, J_n\!\left(\frac{j_{nm}'\rho}{a}\right)\int_0^a t f_n(t)\, J_n\!\left(\frac{j_{nm}'t}{a}\right)\mathrm{d}t = 0.$$

The equation is automatically satisfied when $m = 0$, $n \neq 0$. In the other cases

$$P_n(\kappa_{nm}) = \tfrac{1}{2}\pi j_{nm}'\kappa_{nm}\, a Y_n'(j_{nm}') \int_0^1 t f_n(at)\, J_n(j_{nm}'t)\,\mathrm{d}t. \tag{9}$$

This relation gives the coefficients of an expansion of $f_n(at)$ in a Dini series (Watson 1944). The Dini coefficient f_{nm} is defined by

$$f_{nm} = \frac{2j_{nm}'^2}{(j_{nm}'^2 - n^2)\, J_n^2(j_{nm}')}\int_0^1 t f_n(at)\, J_n(j_{nm}'t)\,\mathrm{d}t \quad (n+m \neq 0), \tag{10}$$

$$f_{00} = 2\delta_{0n}\int_0^1 t f_0(at)\,\mathrm{d}t, \tag{11}$$

where

$$\delta_{mn} = 0 \ \ (n \neq m), \quad = 1 \ \ (n = m).$$

Now, with fixed ρ, $t^{\frac{1}{2}} p_n(at, \rho)$ is of bounded variation and continuous for $0 \leqslant t \leqslant 1$. Hence the Dini series (multiplied by $t^{\frac{1}{2}}$) for $p_n(at, \rho)$ is uniformly convergent to $t^{\frac{1}{2}} p_n(at, \rho)$ in the closed interval $(0, 1)$. Thus it may be multiplied by $t^{\frac{1}{2}} f_n(at)$ and integrated term by term over any range in $(0, 1)$. This process gives the same result as when the Dini series associated with $f_n(at)$ is multiplied by $tp_n(at, \rho)$ and integrated term by term. Hence

$$\int_0^1 tf_n(at)\, p_n(at,\rho)\, \mathrm{d}t = f_{00}\int_0^1 tp_n(at,\rho)\,\mathrm{d}t + \sum_{m=1}^{\infty} f_{nm}\int_0^1 tp_n(at,\rho)\, J_n(j'_{nm}t)\,\mathrm{d}t. \tag{12}$$

We derive from (9), (10) and (11)

$$f_{nm} = \frac{2j'^2_{nm}P_n(\kappa_{nm})}{a\kappa_{nm}(j'^2_{nm}-n^2)\,J_n(j'_{nm})} \quad (n+m \neq 0), \tag{13}$$

$$f_{00} = 2\delta_{0n}P_0(\mathrm{i}k)/a\mathrm{i}k, \tag{14}$$

since $$J_n(j'_{nm})\, Y'_n(j'_{nm}) = 2/\pi j'_{nm}.$$

Equation (12) can be used to evaluate the integral in (7). The evaluation requires the formulae

$$\int^t t\mathscr{C}_\mu(lt)\,\overline{\mathscr{C}}_\mu(mt)\,\mathrm{d}t = \frac{t}{l^2-m^2}\{l\mathscr{C}_{\mu+1}(lt)\,\overline{\mathscr{C}}_\mu(mt) - m\mathscr{C}_\mu(lt)\,\overline{\mathscr{C}}_{\mu+1}(mt)\},$$

where $\mathscr{C}_\mu$, $\overline{\mathscr{C}}_\mu$ are any two cylinder functions of order μ (Lommel 1879) and

$$J_{n+1}(z)\, Y_n(z) - Y_{n+1}(z)\, J_n(z) = 2/\pi z$$

(Lommel 1871). The resulting expression for $U_n(\rho, s)$ is given by

$$U_n(\rho,s) = \frac{P_n(s)\, J_n(\kappa\rho)}{\kappa J'_n(\kappa a)} + s\sum_{m=0}^{\infty}\frac{f_{nm}J_n(j'_{nm}\rho/a)}{s^2-\kappa^2_{nm}} \quad (\rho < a). \tag{15}$$

Conditions (iii) and (v) imply that

$$\lim_{\rho\to a+0}\int_0^\infty u_n(\rho,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z + \frac{u_n^{(0)}(a)-u_n^{(1)}(a)}{s+\mathrm{i}k\cos\theta} = \lim_{\rho\to a-0}\int_0^\infty u_n(\rho,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z = \lim_{\rho\to a-0} U_n(\rho,s), \tag{16}$$

since $u_n = 0$ in $z < 0$, $\rho < a$. Hence

$$\lim_{\rho\to a+0} U_n(\rho,s) = N_n(s) + \lim_{\rho\to a+0}\int_0^\infty u_n(\rho,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z = N_n(s) + \lim_{\rho\to a-0} U_n(\rho,s) - \frac{v_n^{(1)}(a)}{s+\mathrm{i}k\cos\theta},$$

where $$N_n(s) = \lim_{\rho\to a+0}\int_{-\infty}^0 u_n(\rho,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z$$

and $$v_n^{(1)}(a) = u_n^{(0)}(a) - u_n^{(1)}(a) = -2\mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}n\pi}/\{\pi ka\sin\theta H_n^{(2)\prime}(ka\sin\theta)\}.$$

The function $N_n(s)$ is analytic in the negative half-plane.

Substitution for U_n from (8) and (15) gives

$$N_n(s) = \frac{2P_n(s)}{a\kappa^2K^{(n)}(s)} + \frac{v_n^{(1)}(a)}{s+\mathrm{i}k\cos\theta} - s\sum_{m=0}^{\infty}\frac{f_{nm}J_n(j'_{nm})}{s^2-\kappa^2_{nm}}, \tag{17}$$

where $$K^{(n)}(s) = -\pi\mathrm{i}J'_n(\kappa a)\, H_n^{(2)\prime}(\kappa a).$$

It is shown in the appendix that $K^{(n)}(s)$ can be written as $K_P^{(n)}(s)/K_N^{(n)}(s)$, where $K_P^{(n)}$, $K_N^{(n)}$ have no zeros or singularities in the positive and negative half-planes respectively. Further

$K_P^{(n)}(s)\,K_N^{(n)}(-s)=1$, from (A 3), and $K_P^{(n)}(s)\sim(as)^{-\frac{1}{2}}$ as $|s|\to\infty$ in the positive half-plane (A 4).

Equation (17) can now be rewritten as

$$\frac{(s-\mathrm{i}k)\,N_n(s)}{K_N^{(n)}(s)}-\frac{v_n^{(1)}(a)}{s+\mathrm{i}k\cos\theta}\left\{\frac{s-\mathrm{i}k}{K_N^{(n)}(s)}+\frac{\mathrm{i}k(1+\cos\theta)}{K_N^{(n)}(-\mathrm{i}k\cos\theta)}\right\}$$
$$+\sum_{m=0}^{\infty}\frac{f_{nm}(s-\mathrm{i}k)\,J_n(j'_{nm})}{2(s-\kappa_{nm})\,K_N^{(n)}(s)}+\sum_{m=0}^{\infty}\frac{f_{nm}J_n(j'_{nm})}{2(s+\kappa_{nm})}\left\{\frac{s-\mathrm{i}k}{K_N^{(n)}(s)}+\frac{\kappa_{nm}+\mathrm{i}k}{K_N^{(n)}(-\kappa_{nm})}\right\}$$
$$=\frac{2P_n(s)}{a(s+\mathrm{i}k)\,K_P^{(n)}(s)}-\frac{\mathrm{i}k(1+\cos\theta)\,v_n^{(1)}(a)}{(s+\mathrm{i}k\cos\theta)\,K_N^{(n)}(-\mathrm{i}k\cos\theta)}+\sum_{m=0}^{\infty}\frac{(\kappa_{nm}+\mathrm{i}k)f_{nm}J_n(j'_{nm})}{2(s+\kappa_{nm})\,K_N^{(n)}(-\kappa_{nm})}. \tag{18}$$

Since $P_n(s)=O(s^{-\frac{3}{2}})$ as $|s|\to\infty$ in the positive half-plane (on account of condition (v)), $f_{nm}=O(m^{-\frac{7}{6}})$ as $m\to\infty$. Thus the infinite series in (18) are absolutely convergent for all s except $\pm\kappa_{nm}$.

The left-hand side of (18) is analytic in the negative half-plane, whereas the right-hand side is analytic in the positive half-plane. Both sides have the strip $-k_i\cos\theta<\sigma<k_i$ in common and hence must equal an integral function. As $|s|\to\infty$ in the negative half-plane the left-hand side is $o(s^{\frac{1}{2}})$. As $|s|\to\infty$ in the positive half-plane the right-hand side is $o(s^{-\frac{1}{2}})$. Hence, by the extension of Liouville's theorem, the integral function must be the constant zero. Therefore

$$\frac{2P_n(s)}{a(s+\mathrm{i}k)\,K_P^{(n)}(s)}=\frac{\mathrm{i}k(1+\cos\theta)\,v_n^{(1)}(a)}{(s+\mathrm{i}k\cos\theta)\,K_N^{(n)}(-\mathrm{i}k\cos\theta)}-\sum_{m=0}^{\infty}\frac{(\kappa_{nm}+\mathrm{i}k)f_{nm}J_n(j'_{nm})}{2(s+\kappa_{nm})\,K_N^{(n)}(-\kappa_{nm})}$$
$$=\frac{\mathrm{i}k(1+\cos\theta)\,v_n^{(1)}(a)}{(s+\mathrm{i}k\cos\theta)\,K_N^{(n)}(-\mathrm{i}k\cos\theta)}-\frac{2P_0(\mathrm{i}k)\,\delta_{0n}}{a(s+\mathrm{i}k)\,K_N^{(n)}(-\mathrm{i}k)}$$
$$-\sum_{m=1}^{\infty}\frac{(\kappa_{nm}+\mathrm{i}k)\,j'^2_{nm}P_n(\kappa_{nm})}{a\kappa_{nm}(j'^2_{nm}-n^2)\,(s+\kappa_{nm})\,K_N^{(n)}(-\kappa_{nm})},$$

after a use of (13) and (14).

Let

$$\alpha_n=\tfrac{1}{2}\pi ak(1+\cos\theta)\,v_n^{(1)}(a)\,K_P^{(n)}(\mathrm{i}k\cos\theta),$$
$$\alpha_n\alpha_{nm}=\frac{(\kappa_{nm}+\mathrm{i}k)\,\pi j'^2_{nm}P_n(\kappa_{nm})}{2\mathrm{i}\kappa_{nm}(j'^2_{nm}-n^2)\,K_N^{(n)}(-\kappa_{nm})}\quad(m\geqslant 1),$$
$$\alpha_0\alpha_{00}=-\pi\mathrm{i}P_0(\mathrm{i}k)/K_N^{(0)}(-\mathrm{i}k),$$
$$\alpha_{n0}=0\quad(n\neq 0).$$

Then the equation for $P_n(s)$ becomes

$$\frac{-\pi\mathrm{i}P_n(s)}{(s+\mathrm{i}k)\,K_P^{(n)}(s)}=\frac{\alpha_n}{s+\mathrm{i}k\cos\theta}-\frac{\delta_{0n}\alpha_0\alpha_{00}}{s+\mathrm{i}k}-\sum_{m=1}^{\infty}\frac{\alpha_n\alpha_{nm}}{s+\kappa_{nm}}. \tag{19}$$

The constants α_{nm} may be determined by putting $s=\kappa_{nm}$ $(m=0,1,\ldots)$ in (19). This process leads to the following equations for α_{nm}:

$$\alpha'_{nr}\alpha_{nr}=\frac{1}{\kappa_{nr}+\mathrm{i}k\cos\theta}-\frac{\delta_{0n}\alpha_{00}}{\kappa_{nr}+\mathrm{i}k}-\sum_{m=1}^{\infty}\frac{\alpha_{nm}}{\kappa_{nr}+\kappa_{nm}}\quad(r=0,1,\ldots), \tag{20}$$

where

$$\alpha'_{nr}=\frac{2\kappa_{nr}(j'^2_{nr}-n^2)}{\{j'_{nr}(\kappa_{nr}+\mathrm{i}k)\,K_P^{(n)}(\kappa_{nr})\}^2}\quad(n+r\neq 0),$$
$$\alpha'_{00}=1/2\mathrm{i}k\{K_P^{(0)}(\mathrm{i}k)\}^2.$$

We require a solution of our problem such that $P_n(s) = O(s^{-\frac{2}{3}})$ as $|s|\to\infty$ (condition (v)), which means that we need a solution of the equations (20) such that $\alpha_{nm} \sim m^{-\frac{7}{6}}$ as $m\to\infty$. In addition, by deforming the contour to the right we may show that

$$\int_{c-i\infty}^{c+i\infty} \frac{P_n(s)\,ds}{(s+ik)\,K_P^{(n)}(s)} = 0$$

where $-k_i\cos\theta < c < k_i$. Hence we must have

$$\delta_{0n}\alpha_{00} + \sum_{m=1}^{\infty}\alpha_{nm} = 1. \tag{21}$$

The function $U_n(\rho, s)$ is known from (8) when the function $P_n(s)$ is given, and hence, by the use of the Mellin inverse, the function $u_n(\rho, z)$ can be obtained when $P_n(s)$ is known. The form of the solution is now known, being given essentially by (19), and the solution is known precisely as soon as the equations (20) are solved for the constants α_{nm}.

It may be remarked that we can obtain the solution for the hollow semi-infinite cylinder merely by putting $\alpha_{nm} = 0$ (all n and m) in (19).

2. The boundary condition $\partial u/\partial \nu = 0$ when the angle of incidence is π

In this section we take the incident wave to be e^{ikz}. The total field is assumed to be

$$\begin{array}{lll} e^{ikz}+u & \text{in} & \rho > a, \\ e^{ikz}+e^{-ikz}+u & \text{in} & \rho < a,\ z>0, \\ 0 & \text{in} & \rho<a,\ z<0. \end{array}$$

Then u is independent of ϕ and satisfies conditions (i) to (v), except that in (iv) $u \sim e^{-ikz}/z$ as $z\to\infty$ in $\rho < a$. Consequently the analysis of the preceding section with $n = 0$ may be used as far as equation (15). Equation (16) will remain valid provided that $\theta = 0$ and

$$u_n^{(0)}(a) - u_n^{(1)}(a) = -1.$$

With this modification the analysis can be carried through as before. Hence

$$U(\rho, s) = \frac{P(s)\,H_0^{(2)}(\kappa\rho)}{\kappa H_0^{(2)\prime}(\kappa a)} \quad \text{in} \quad \rho > a$$

$$= \frac{P(s)\,J_0(\kappa\rho)}{\kappa J_0'(\kappa a)} + s\sum_{m=0}^{\infty}\frac{h_m J_0(j_{0m}'\rho/a)}{s^2-\kappa_{0m}^2} \quad \text{in} \quad \rho<a,$$

where $$U(\rho,s) = \int_{-\infty}^{\infty} u(\rho,z)\,e^{-sz}\,dz, \quad h_m = \frac{2P(\kappa_{0m})}{a\kappa_{0m}J_0(\kappa_{0m})} \quad (m \neq 0),$$

$$h_0 = 2P(ik)/aik), \tag{22}$$

and $$\frac{-\pi i P(s)}{\beta_0'(s+ik)\,K_P^{(0)}(s)} = \frac{1-\beta_0}{s+ik} - \sum_{m=1}^{\infty}\frac{\beta_m}{s+\kappa_{0m}},$$

where $$\beta_0' = -\pi ak K_P^{(0)}(ik), \tag{23}$$

$$\beta_0'\beta_m = -\tfrac{1}{2}\pi i(ik+\kappa_{0m})\,P(\kappa_{0m})\,K_P^{(0)}(\kappa_{0m})/\kappa_{0m}. \tag{24}$$

Equations for the constants h_m in terms of the value of u on the end may be obtained from (10) and (11) by replacing f_{0m} by h_m.

The equations for the constants β_m are

$$\frac{2\kappa_{0r}\beta_r}{\{(\kappa_{0r}+\mathrm{i}k)\,K_p^{(0)}(\kappa_{0r})\}^2}=\frac{1-\beta_0}{\kappa_{0r}+\mathrm{i}k}-\sum_{m=1}^{\infty}\frac{\beta_m}{\kappa_{0r}+\kappa_{0m}}\quad(r=0,1,\ldots). \tag{25}$$

The solution is now in the same form as that for the preceding problem.

3. The boundary condition $u=0$

In the discussion of the boundary condition $u=0$ we shall restrict the analysis to those cases in which the angle of incidence is not 0 or π, i.e. we shall take the incident wave to be $u^{(0)}$.

Let
$$u^{(2)}=\mathrm{e}^{-\mathrm{i}kz\cos\theta}\sum_{n=-\infty}^{\infty}u_n^{(2)}(\rho)\,\mathrm{e}^{\mathrm{i}n\phi},$$
where
$$u_n^{(2)}(\rho)=\mathrm{e}^{-\frac{1}{2}\mathrm{i}n\pi}\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)}(k\rho\sin\theta).$$
Then $u^{(0)}=u^{(2)}$ on $\rho=a$.

Let the total field be given by

$$\begin{aligned}&u^{(0)}-u^{(2)}+u(\rho,\phi,z) &&\text{in}\quad \rho\geqslant a,\\ &u(\rho,\phi,z) &&\text{in}\quad \rho\leqslant a,\ z\geqslant 0,\\ &0 &&\text{in}\quad \rho\leqslant a,\ z\leqslant 0.\end{aligned}$$

Then u satisfies (1) and the conditions

(i)′ $u=0$ on the surface of the rod,

(ii)′ u is continuous across $\rho=a$ for all z except possibly $z=0$,

(iii)′ grad u is continuous except possibly across the surface of the rod,

(iv)′ the same as (iv),

(v)′ $u=O(\varpi^{\frac{2}{3}})$, $|\mathrm{grad}\,u|=O(\varpi^{-\frac{1}{3}})$ as $\varpi\to 0$.

The equation for $U_n(\rho,s)$ is

$$\left.\begin{aligned}\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}U_n}{\mathrm{d}\rho}\right)+\left(\kappa^2-\frac{n^2}{\rho^2}\right)U_n&=0 &&(\rho>a),\\ &=g_n(\rho) &&(\rho<a),\end{aligned}\right\} \tag{26}$$

where $g_n(\rho)=\lim\limits_{z\to+0}\partial u_n/\partial z$.

It follows from (i)′, (ii)′ and (26) that

$$U_n(\rho,s)=R_n(s)\,H_n^{(2)}(\kappa\rho)/H_n^{(2)}(\kappa a),\qquad(\rho\geqslant a) \tag{27}$$

$$=\{1/J_n(\kappa a)\}\{R_n(s)\,J_n(\kappa\rho)+\int_0^a tg_n(t)\,q_n(t,\rho)\,\mathrm{d}t\}\quad(\rho\leqslant a), \tag{28}$$

where
$$\begin{aligned}q_n(t,\rho)&=\tfrac{1}{2}\pi\{J_n(\kappa a)\,Y_n(\kappa\rho)-Y_n(\kappa a)\,J_n(\kappa\rho)\}\,J_n(\kappa t)\quad(0\leqslant t\leqslant\rho)\\ &=\tfrac{1}{2}\pi\{J_n(\kappa a)\,Y_n(\kappa t)-Y_n(\kappa a)\,J_n(\kappa t)\}\,J_n(\kappa\rho)\quad(\rho\leqslant t\leqslant a).\end{aligned}$$

The function $R_n(s)$ is analytic in the positive half-plane. $U_n(\rho,s)$ must be analytic in the positive half-plane and therefore (28) cannot have poles at zeros of $J_n(\kappa a)$. Let $J_n(j_{nm})=0$, where $j_{nm}>j_{n,m-1}$ and $j_{n0}=0$. We define $j_{00}=0$ although it is not a zero of J_0. Write $\lambda_{nm}^2=j_{nm}^2/a^2-k^2$ and define λ_{nm}, when $k_i=0$, to be positive when real and positive imaginary otherwise. Then
$$\mathscr{R}(\lambda_{nm})\geqslant k_i,\quad \mathscr{I}(\lambda_{nm})\leqslant k_r,$$
the equality holding only when $m=0$.

Hence $\quad R_n(\lambda_{nm})\, J_n(j_{nm}\rho/a) - \tfrac{1}{2}\pi Y_n(j_{nm})\, J_n(j_{nm}\rho/a) \int_0^a t g_n(t)\, J_n(j_{nm}t/a)\, \mathrm{d}t = 0,$

when $m \geqslant 0$, $n > 0$, or $m \neq 0$, $n = 0$. The equation is satisfied identically when $m = 0$, $n > 0$. Consequently

$$R_n(\lambda_{nm}) = \tfrac{1}{2}\pi a^2 Y_n(j_{nm}) \int_0^1 t g_n(at)\, J_n(j_{nm}t)\, \mathrm{d}t \quad (m = 1, 2, \ldots).$$

This relation gives the coefficients in the Fourier–Bessel expansion of $g_n(at)$ (Watson 1944). By an argument similar to that of § 1 it can now be shown that

$$U_n(\rho, s) = \frac{R_n(s)\, J_n(\kappa\rho)}{J_n(\kappa a)} + \sum_{m=1}^{\infty} \frac{g_{nm} J_n(j_{nm}\rho/a)}{s^2 - \lambda_{nm}^2} \quad (\rho \leqslant a), \tag{29}$$

where

$$g_{nm} = \frac{2}{J_{n+1}^2(j_{nm})} \int_0^1 t g_n(at)\, J_n(j_{nm}t)\, \mathrm{d}t = \frac{2 j_{nm} R_n(\lambda_{nm})}{a^2 J_{n+1}(j_{nm})}.$$

Conditions (iii)′ and (v)′ imply that

$$\lim_{\rho \to a+0} \int_0^{\infty} \frac{\partial u_n}{\partial \rho} \mathrm{e}^{-sz}\, \mathrm{d}z + \frac{u_n^{(0)\prime}(a) - u_n^{(2)\prime}(a)}{s + \mathrm{i}k\cos\theta} = \lim_{\rho \to a-0} \int_0^{\infty} \frac{\partial u_n}{\partial \rho} \mathrm{e}^{-sz}\, \mathrm{d}z = \lim_{\rho \to a-0} U_n',$$

where $U_n' = \partial U_n/\partial\rho$. Hence

$$\lim_{\rho \to a+0} U_n' = N_n'(s) + \lim_{\rho \to a-0} U_n' - v_n^{(2)}(a)/(s + \mathrm{i}k\cos\theta),$$

where

$$N_n'(s) = \lim_{\rho \to a+0} \int_{-\infty}^{0} \frac{\partial u_n}{\partial \rho} \mathrm{e}^{-sz}\, \mathrm{d}z$$

and $\quad v_n^{(2)}(a) = u_n^{(0)\prime}(a) - u_n^{(2)\prime}(a) = 2\mathrm{i}\, \mathrm{e}^{-\frac{1}{2}\mathrm{i}n\pi}/\pi ka \sin\theta H_n^{(2)}(ka\sin\theta).$

$N_n'(s)$ is analytic in the negative half-plane.

It follows from (27) and (29) that

$$N_n'(s) - \frac{v_n^{(2)}(a)}{s + \mathrm{i}k\cos\theta} = -\frac{2R_n(s)}{aL^{(n)}(s)} - \sum_{m=1}^{\infty} \frac{g_{nm} j_{nm} J_n'(j_{nm})}{a(s^2 - \lambda_{nm}^2)},$$

where

$$L^{(n)}(s) = -\pi\mathrm{i} J_n(\kappa a)\, H_n^{(2)}(\kappa a).$$

It is shown in the appendix that $L^{(n)}(s)$ can be written as $L_P^{(n)}(s)/L_N^{(n)}(s)$, where $L_P^{(n)}$, $L_N^{(n)}$ have no zeros or singularities in the positive and negative half-planes respectively. Further $L_P^{(n)}(s)\, L_N^{(n)}(-s) = 1$ (A 11) and $L_P^{(n)}(s) \sim (as)^{-\frac{1}{2}}$ as $|s| \to \infty$ in the positive half-plane (A 12). In a similar way to that in § 1 it is found that

$$\frac{2R_n(s)}{aL_P^{(n)}(s)} = \frac{v_n^{(2)}(a)}{(s + \mathrm{i}k\cos\theta)\, L_N^{(n)}(-\mathrm{i}k\cos\theta)} + \sum_{m=1}^{\infty} \frac{g_{nm} j_{nm} J_n'(j_{nm})}{2a\lambda_{nm}(s + \lambda_{nm})\, L_N^{(n)}(-\lambda_{nm})}$$
$$= \frac{v_n^{(2)}(a)}{(s + \mathrm{i}k\cos\theta)\, L_N^{(n)}(-\mathrm{i}k\cos\theta)} - \sum_{m=1}^{\infty} \frac{j_{nm}^2 R_n(\lambda_{nm})}{a^3 \lambda_{nm} L_N^{(n)}(-\lambda_{nm})\, (s + \lambda_{nm})}.$$

Let

$$\gamma_n = -\tfrac{1}{2}\mathrm{i}\pi a v_n^{(2)}(a)\, L_P^{(n)}(\mathrm{i}k\cos\theta), \tag{30}$$

$$\gamma_n \gamma_{nm} = \frac{\pi j_{nm}^2 R_n(\lambda_{nm})}{2\mathrm{i}a^2 \lambda_{nm} L_N^{(n)}(-\lambda_{nm})}.$$

The equation for $R_n(s)$ becomes

$$-\frac{\pi \mathrm{i} R_n(s)}{L_P^{(n)}(s)} = \frac{\gamma_n}{s+\mathrm{i}k\cos\theta} - \sum_{m=1}^{\infty} \frac{\gamma_n \gamma_{nm}}{s+\lambda_{nm}}, \tag{31}$$

and the equations to determine γ_{nm} are

$$\frac{2a^2\lambda_{nr}\gamma_{nr}}{\{j_{nr}L_P^{(n)}(\lambda_{nr})\}^2} = \frac{1}{\lambda_{nr}+\mathrm{i}k\cos\theta} - \sum_{m=1}^{\infty} \frac{\gamma_{nm}}{\lambda_{nr}+\lambda_{nm}} \quad (r = 1, 2, \ldots). \tag{32}$$

We require a solution such that $R_n(s) = O(s^{-\frac{4}{3}})$ as $|s| \to \infty$ and hence a solution of (32) such that $\gamma_{nm} = O(m^{-\frac{7}{6}})$ as $m \to \infty$. Also

$$\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{R_n(s)\,\mathrm{d}s}{L_P^{(n)}(s)} = 0,$$

and so

$$\sum_{m=1}^{\infty} \gamma_{nm} = 1. \tag{33}$$

It is to be noted that the solution for the hollow semi-infinite cylinder may be obtained by putting $\gamma_{nm} = 0$ (all n and m).

4. The distant field

When the boundary condition is $\partial u/\partial \nu = 0$ the field in the nth mode is given by

$$u_n(\rho, z) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} U_n(\rho, s)\,\mathrm{e}^{sz}\,\mathrm{d}s$$

$$= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{P_n(s)\,H_n^{(2)}(\kappa\rho)}{\kappa H_n^{(2)\prime}(\kappa a)}\,\mathrm{e}^{\mathrm{i}\kappa\rho}\,\mathrm{e}^{sz-\mathrm{i}\kappa\rho}\,\mathrm{d}s \quad \text{in} \quad \rho > a,$$

where $P_n(s)$ is obtained from (19). The field at large distances from the origin may be obtained by the saddle-point method of approximation.

Let $z = r\cos\psi$, $\rho = r\sin\psi$, where $0 < \psi < \pi$. The saddle-point of $sz - \mathrm{i}\kappa\rho$ occurs at $s = -\mathrm{i}k\cos\psi$. The contour of integration may be deformed into the curve of steepest descent through the saddle-point, care being taken to add the contributions of any poles which are passed over. When ψ is not near 0 or π it is easy to show that the curve of steepest descent does not pass near $s = \pm\mathrm{i}k$. Hence, as $kr \to \infty$, we may take

$$H_n^{(2)}(\kappa\rho)\,\mathrm{e}^{\mathrm{i}\kappa\rho} = \left(\frac{2}{\pi\kappa\rho}\right)^{\frac{1}{2}} \mathrm{e}^{\pi\mathrm{i}(\frac{1}{2}n+\frac{1}{4})}$$

on this curve. Consequently, as $kr \to \infty$, the contribution from the curve of steepest descent is

$$\frac{\mathrm{i}P_n(-\mathrm{i}k\cos\psi)}{\pi k\sin\psi H_n^{(2)\prime}(ak\sin\psi)} \frac{\mathrm{e}^{-\mathrm{i}kr+\frac{1}{2}\mathrm{i}n\pi}}{r}, \tag{34}$$

when ψ is not near 0, θ or π.

When $\psi < \theta$ the contour is deformed over the simple pole $s = -\mathrm{i}k\cos\theta$. The contribution from this pole is $u_n^{(1)}\,\mathrm{e}^{-\mathrm{i}kz\cos\theta}$ which removes the reflected wave in the region $\psi < \theta$. When ψ is near θ the above approximation breaks down because the saddle-point is near a pole. This difficulty may be overcome by the method used previously by the author (Jones 1953 *a*) or by the more general method of Clemmow (1950) but will not be considered in detail here.

There remains the question of the possible contributions from the poles due to the zeros of $H_n^{(2)\prime}(\kappa a)$. It can be shown that $H_n^{(2)\prime}(z)$ has no zeros for $0 \geqslant \arg z \geqslant -\pi$. Now the curve of steepest descent always lies in a region where $0 \geqslant \arg \kappa \geqslant -\pi$, since it crosses from one sheet to another of the Riemann surface of $(s^2+k^2)^{\frac{1}{2}}$ when the contrary would occur. The contour of integration can be deformed into the curve of steepest descent without leaving a region in which $0 \geqslant \arg \kappa \geqslant -\pi$ and hence there is no contribution from the zeros of $H_n^{(2)\prime}(\kappa a)$.

The distant field, when the boundary condition is $u=0$, is

$$\frac{\mathrm{i}R_n(-\mathrm{i}k\cos\psi)}{\pi H_n^{(2)}(ak\sin\psi)}\frac{\mathrm{e}^{-\mathrm{i}kr+\frac{1}{2}\mathrm{i}n\pi}}{r}, \tag{35}$$

together with a term which removes the reflected wave in $\psi<\theta$.

It has already been remarked that the field consists of that due to a hollow semi-infinite cylinder together with a field due to the end of the cylinder obtained from the constants α_{nm} and γ_{nm} respectively. When account of this is taken in (34) and (35) we see that the distant field due to the end in the nth mode is

$$\left\{\frac{\delta_{0n}\alpha_0\alpha_{00}}{\mathrm{i}k(1-\cos\psi)}+\sum_{m=1}^{\infty}\frac{\alpha_n\alpha_{nm}}{\kappa_{nm}-\mathrm{i}k\cos\psi}\right\}\frac{\mathrm{i}(1-\cos\psi)\,K_P^{(n)}(-\mathrm{i}k\cos\psi)}{\pi^2\sin\psi H_n^{(2)\prime}(ka\sin\psi)}\frac{\mathrm{e}^{-\mathrm{i}kr+\frac{1}{2}\mathrm{i}n\pi}}{r} \tag{36}$$

for the boundary condition $\partial u/\partial\nu=0$ and

$$\frac{L_P^{(n)}(-\mathrm{i}k\cos\psi)}{\pi^2 H_n^{(2)}(ka\sin\psi)}\frac{\mathrm{e}^{-\mathrm{i}kr+\frac{1}{2}\mathrm{i}n\pi}}{r}\sum_{m=1}^{\infty}\frac{\gamma_n\gamma_{nm}}{\lambda_{nm}-\mathrm{i}k\cos\psi} \tag{37}$$

for the boundary conditions $u=0$. It is shown in § 6 that we can form equivalent expressions which are stationary for small variations of the constants α_{nm} and γ_{nm} about their correct values.

5. The average pressure amplitude and the scattering coefficient for the boundary condition $\partial u/\partial\nu=0$

We deal first with the case when the angle of incidence is π. We assume that the pressure amplitude in the incident wave is unity. Then the total pressure on the end of the rod is

$$2\pi\int_0^a t(u+2)_{z=0}\,\mathrm{d}t,$$

and hence the average pressure amplitude, i.e. (total pressure)/(end area) is the modulus of

$$\frac{2}{a^2}\int_0^a t(u+2)_{z=0}\,\mathrm{d}t.$$

It follows from (11), (22), (23) and (24) that the average pressure amplitude is $2\,|\,1-\beta_0\,|$.

The constant β_0 is obtained by solving the equations (25). This has been carried out approximately by assuming successively

$$\text{(i)}\ \beta_m=0\ \ (m>0),\quad \text{(ii)}\ \beta_m=0\ \ (m>1)\quad \text{and}\quad \text{(iii)}\ \beta_m=0\ \ (m>2).\dagger$$

† It may be remarked that the boundary condition on the sides of the rod is satisfied whatever values are taken for the β_m. Approximation for the β_m implies approximation to the boundary condition on the end of the rod.

The average pressure amplitude supplied by these three approximations is plotted against ka in figure 1 for the range $0 \leqslant ka \leqslant 10$. The values of $K_p^{(0)}(s)$ necessary for this computation are given in table 2, p. 523.

All three approximations give the same qualitative behaviour for the average pressure amplitude—a steady rise from 1 to a maximum near $ka = 2{\cdot}5$, followed by an oscillation about the value of 2. The second and third approximations differ by so little (less than 1 % over most of the range) that it is reasonable to suppose that they give a fairly accurate estimate of the amplitude. Thus the maximum pressure amplitude occurring is 2·18 at $ka = 2{\cdot}4$.

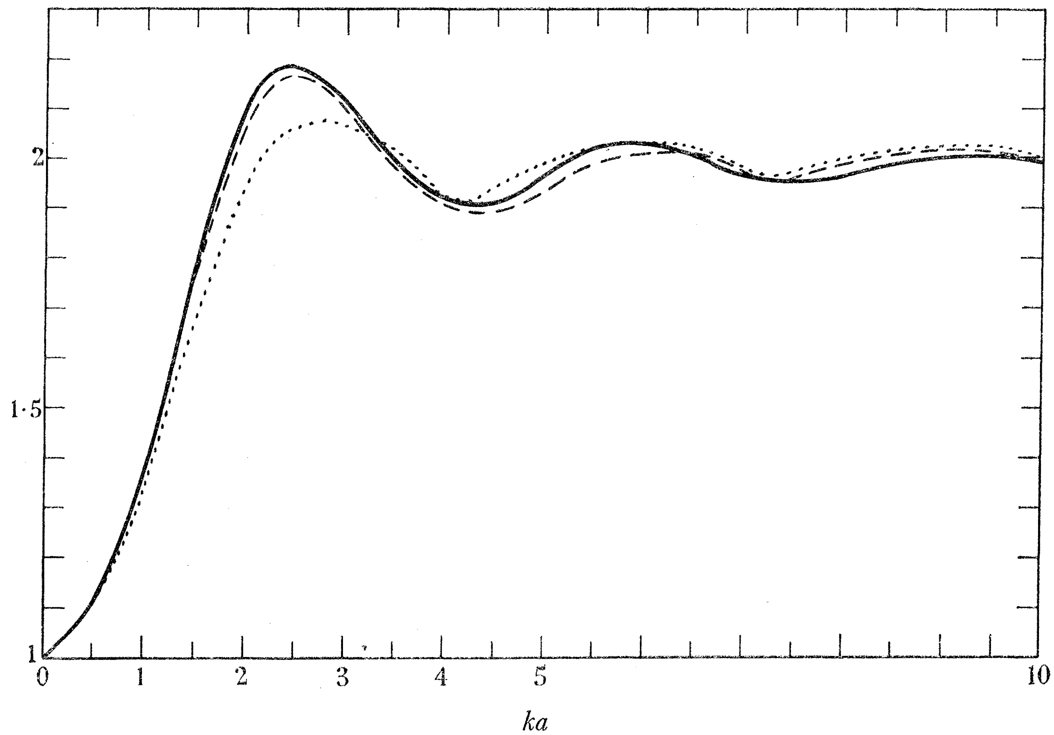

FIGURE 1. The average pressure amplitude on the end of the rod when the angle of incidence is π. Approximations: ...., first; ----, second; ——, third.

In § 6 a variational expression is obtained for β_0. It would be possible to improve the above approximation by putting (say) $\beta_m = 0$ $(m > 2)$ in the right-hand side of equations (25) and substituting the values of β_m (all m) so obtained in the variational expression. However, in view of the close agreement between the approximations above, such a process has not been attempted.

To determine the scattering coefficient let $w(\rho, \phi, z)$ be the total field at any point when the incident field is e^{ikz}. Then the energy scattered by the rod is

$$C\int_S \{(w - e^{ikz}) \operatorname{grad} (w^* - e^{-ikz}) - (w^* - e^{-ikz}) \operatorname{grad} (w - e^{ikz})\} . d\mathbf{S},$$

where C is a constant, the asterisk denotes a complex conjugate and S is the surface of a large sphere at infinity apart from the portion removed by the rod, $d\mathbf{S}$ being along the normal. Since w and e^{ikz} satisfy $\nabla^2 w + k^2 w = 0$ the integral may be converted, by the divergence theorem, into one over the surface of the rod. A use of the fact that $\partial w/\partial\rho = 0$ on $\rho = a$, $z<0$ then shows that the scattered energy is

$$\mathrm{i}kC\int_0^a\int_0^{2\pi}\{2\mathscr{R}w(\rho,\phi,0)-2\}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = 2\pi\mathrm{i}kC\int_0^a(2\mathscr{R}u+2)\,\rho\,\mathrm{d}\rho$$
$$= 2\pi\mathrm{i}kCa^2(1-2\mathscr{R}\beta_0).$$

The incident energy per unit area is $2\mathrm{i}kC$. Hence we have for the scattering cross-section c_1

$$c_1 = \frac{\text{scattered energy}}{\text{incident energy per unit area}}$$
$$= \pi a^2(1-2\mathscr{R}\beta_0);$$

the scattering coefficient $c_0 = c_1/\pi a^2$ is given by

$$c_0 = 1-2\mathscr{R}\beta_0.$$

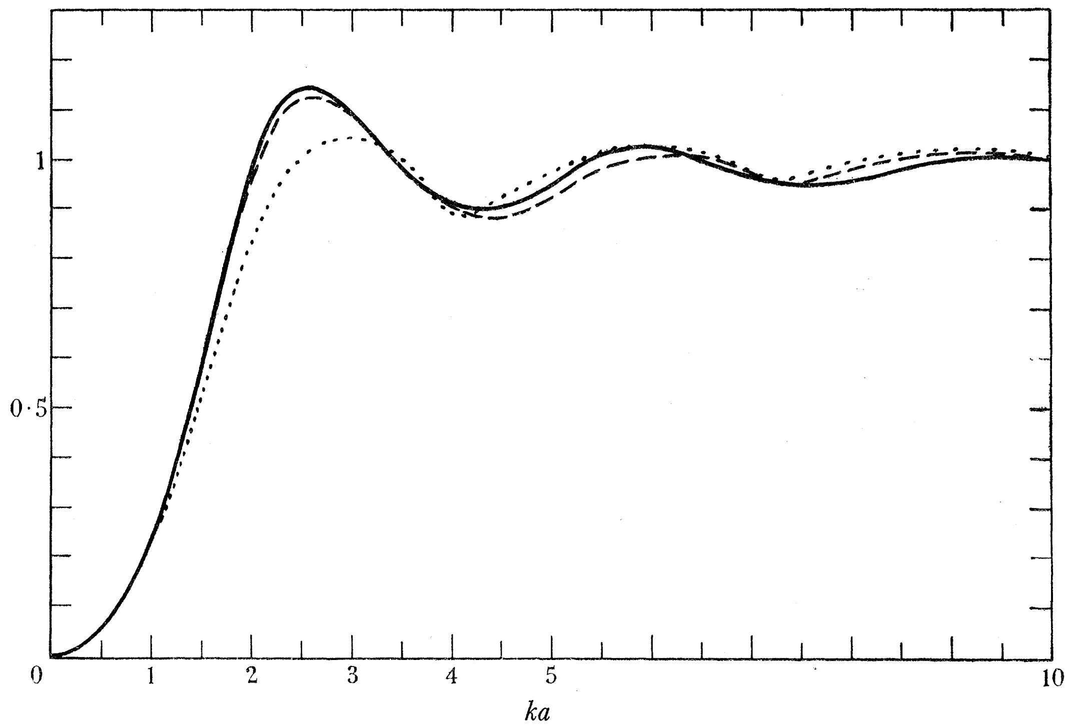

FIGURE 2. The scattering coefficient when the angle of incidence is π. Approximations: . . . ., first; – – – –, second; ——, third.

The scattering coefficient is plotted in figure 2 against ka. It exhibits the usual shape associated with diffraction phenomena, having a maximum of 1·14 at $ka = 2{\cdot}5$.

For small ka we can estimate c_1 and c_0 by using the first approximate formula for β_0. (All three approximations agree in the range $0 \leqslant ka \leqslant 0{\cdot}5$.) This gives

$$c_0 = \mathscr{R}2[1+\{K_P^{(0)}(\mathrm{i}k)\}^2]^{-1}-1$$
$$= \frac{1-|\{K_P^{(0)}(\mathrm{i}k)\}^2|^2}{1+2\mathscr{R}\{K_P^{(0)}(\mathrm{i}k)\}^2+|\{K_P^{(0)}(\mathrm{i}k)\}^2|^2}$$
$$\approx \tfrac{1}{4}[1-|\{K_P^{(0)}(\mathrm{i}k)\}^2|^2],$$

since $\{K_p^{(0)}(\mathrm{i}k)\}^2 \approx 1$ when $ka \ll 1$ (equation (A 9)). Now $K_p^{(0)}(\mathrm{i}k) = \mathrm{e}^{-\alpha I_{\alpha,0}}$, where $\alpha = ka$, from (51) and hence $|\{K_p^{(0)}(\mathrm{i}k)\}^2|^2 = \mathrm{e}^{-2\alpha \mathscr{R} I_{\alpha,0}}$. Since

$$\mathscr{R} I_{\alpha,0} = I'_{\alpha,0} = \tfrac{1}{4}\alpha\{1+o(1)\}$$

as $\alpha \to 0$ from (60) it follows that $$c_0 \approx \tfrac{1}{4}\alpha^2 = \tfrac{1}{4}(ka)^2 \tag{38}$$

for small ka. In fact equation (38) for c_0 agrees well with the numerical values of c_0 for $ka < 2$.

We now consider the average pressure amplitude on the end of the rod when the angle of incidence is not π. Let χ_1 be the total field produced when an incident plane wave travelling in the direction of the unit vector $\mathbf{n}_0$ falls on the rod. Let χ_2 be the total field produced when the wave $\mathrm{e}^{-\mathrm{i}kz}$ is energized at $z = -\infty$ inside a hollow semi-infinite cylinder occupying $\rho = a$, $z \leqslant 0$. Then

$$\int_{S+S_1+S_2} (\chi_1 \operatorname{grad} \chi_2 - \chi_2 \operatorname{grad} \chi_1) \,.\, \mathrm{d}\mathbf{S} = 0,$$

where S is the surface of the sphere $r = R$ apart from the portion removed by the rod, S_1 the surface $\rho = a$, $z \leqslant 0$ and S_2 the surface $\rho \leqslant a$, $z = 0$. Since $\partial\chi_1/\partial\nu = \partial\chi_2/\partial\nu = 0$ on S_1 we obtain

$$\int_{S+S_2} (\chi_1 \operatorname{grad} \chi_2 - \chi_2 \operatorname{grad} \chi_1) \,.\, \mathrm{d}\mathbf{S} = 0.$$

On S
$$\chi_1 \sim \mathrm{e}^{-\mathrm{i}k\mathbf{R}.\mathbf{n}_0} + A(\mathbf{n})\, \mathrm{e}^{-\mathrm{i}kR}/R,$$
$$\chi_2 \sim B(\mathbf{n})\, \mathrm{e}^{-\mathrm{i}kR}/R$$

as $R \to \infty$ where $\mathbf{n}$ is a unit vector in the direction of the point of observation and $\mathbf{R} = R\mathbf{n}$. Hence

$$\lim_{R\to\infty} \int_S (\chi_1 \operatorname{grad} \chi_2 - \chi_2 \operatorname{grad} \chi_1) \,.\, \mathrm{d}\mathbf{S} = -\lim_{R\to\infty} \iint \mathrm{i}k(1-\mathbf{n}\,.\,\mathbf{n}_0)\, B(\mathbf{n})\, \mathrm{e}^{-\mathrm{i}kR - \mathrm{i}k\mathbf{R}.\mathbf{n}_0} R \sin\theta \,\mathrm{d}\theta\, \mathrm{d}\phi.$$

A use of a lemma proved elsewhere (Jones 1952*c*) now shows that

$$\lim_{R\to\infty} \int_S (\chi_1 \operatorname{grad} \chi_2 - \chi_2 \operatorname{grad} \chi_1) \,.\, \mathrm{d}\mathbf{S} = -4\pi B(-\mathbf{n}_0).$$

Inside the hollow cylinder

$$\chi_2 = \mathrm{e}^{-\mathrm{i}kz} + R'\,\mathrm{e}^{\mathrm{i}kz} + \sum_{m=1} a_m J_0(j'_{0m}\rho/a) \exp\{-\mathrm{i}z(j'^2_{0m}/a^2 - k^2)^{\frac{1}{2}}\}$$

and hence

$$\int_{S_2} (\chi_1 \operatorname{grad} \chi_2 - \chi_2 \operatorname{grad} \chi_1) \,.\, \mathrm{d}\mathbf{S} = -\mathrm{i}\pi a^2 [k(R'-1) f_{00} + \sum_{m=1} a_m (j'^2_{0m}/a^2 - k^2)^{\frac{1}{2}} f_{0m} J_0^2(j'_{0m})]$$

after a use of (10) and (11). Consequently, if we assume that $f_{0m} = 0$ $(m > 0)$, we have the result

$$f_{00} = \frac{4\mathrm{i}B(-\mathbf{n}_0)}{ka^2(R'-1)}.$$

Now the average pressure on the end of the rod is

$$\frac{2}{a^2}\int_0^a t u(t, \phi, 0)\, \mathrm{d}t = f_{00},$$

and hence the average pressure on the end of the rod is

$$\frac{4\mathrm{i}B(-\mathbf{n}_0)}{ka^2(R'-1)}.$$

This result may be converted to a slightly different form by introducing the total field χ_3 which arises when a plane wave travelling in the direction $\mathbf{n}_0$ impinges on the semi-infinite hollow cylinder. As $z \to -\infty$ inside the cylinder, when $ka < j'_{01}$,

$$\chi_3 \sim D(\mathbf{n}_0)\, e^{ikz},$$

together with terms from the non-symmetric modes; it may be shown by an argument similar to that above that

$$4\pi B(-\mathbf{n}_0) = 2ik\pi a^2 D(\mathbf{n}_0).$$

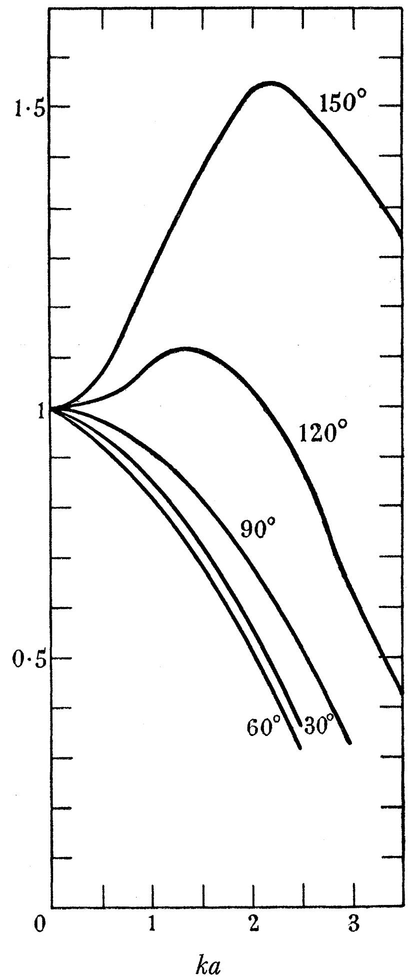

FIGURE 3. The average pressure amplitude on the end of the rod for various angles of incidence.

Hence the average pressure on the end is $2D(\mathbf{n}_0)/(1-R')$. Now

$$R' = -\{K_p^{(0)}(ik)\}^2$$

and so the average pressure is

$$2(1-\beta_0)\, D(\mathbf{n}_0), \tag{39}$$

on account of (25) when we make the assumption that $\beta_m = 0$ $(m \geqslant 1)$.

Thus the assumptions that $f_{0m} = 0$ $(m > 0)$ and that $\beta_m = 0$ $(m > 0)$ imply that the average pressure on the end is the product of the average pressure when the angle of incidence is π and the complex amplitude of the symmetric mode produced inside the semi-infinite hollow cylinder by the same incident plane wave. The assumptions are not likely to give good results outside of the range $ka < 1\cdot 84$, because outside of this range the effect of non-symmetric modes will become important. Nevertheless, the results for an angle of incidence of π suggest that our approximation will give qualitatively accurate information and that the quantitative result will not be in error by more than 10% (considerably less for $ka \leqslant 1$) in the range $ka < 1\cdot 84$. The average pressure when $ka \geqslant 4$ may be estimated by the methods of geometrical optics when the incident wave 'illuminates' the end of the rod.

The average pressure amplitude determined by (39) is plotted against ka in figure 3. It can be seen that when the incident wave 'illuminates' the end of the rod the average pressure amplitude has a maximum greater than 1 which, however, becomes smaller and occurs closer to $ka = 0$ as the angle of incidence decreases from π to $\frac{1}{2}\pi$. For angles of incidence less than $\frac{1}{2}\pi$ no maximum occurs away from $ka = 0$—the amplitude drops steadily as ka increases. Although it has been anticipated above that the approximation is only qualitatively correct for $ka > 1\cdot 84$ it will be found that average pressure amplitude calculated by geometrical optics for $\theta = 120^\circ$, 150° ($ka \geqslant 4$) joins on quite smoothly to the values obtained by our approximation in $0 \leqslant ka \leqslant 3\cdot 5$.

6. Variational expressions

The equations (20), (25) and (32) may all be included in the system

$$b_p(\theta) = T_{pq}\, a_q(\theta), \tag{40}$$

where the repeated suffix implies summation, provided that the symbols are suitably interpreted; for example, for (20)

$$b_p(\theta) = 1/(\kappa_{np} + \mathrm{i}k\cos\theta), \quad a_q(\theta) = \alpha_{nq},$$
$$T_{pq} = \alpha'_{np}\,\delta_{pq} + 1/(\kappa_{np} + \kappa_{nq}).$$

The same quantities may be used for (25) on the understanding that θ and n are zero. Note that T_{pq} is independent of θ.

Let
$$A(\theta, \psi) = a_q(\theta)\, b_q(\pi - \psi);$$

then $A(\theta, \psi)$ gives the series required in (36), (37) and also an expression for β_0 after a use of (25), when the convention in the preceding paragraph is used.

It follows from (40) that

$$\begin{aligned} A(\theta, \psi) &= a_q(\theta)\, T_{qp}\, a_p(\pi - \psi) \\ &= b_p(\theta)\, a_p(\pi - \psi), \quad \text{since} \quad T_{pq} = T_{qp}, \\ &= A(\pi - \psi, \pi - \theta). \end{aligned}$$

It is easily verified that the convergence requirements of the above process are satisfied by the types of solution under consideration. Therefore

$$A(\theta, \psi) = A(\pi - \psi, \pi - \theta) = \frac{a_q(\theta)\, b_q(\pi - \psi)\, a_p(\pi - \psi)\, b_p(\theta)}{a_q(\theta)\, T_{qp}\, a_p(\pi - \psi)}. \tag{41}$$

Now make a small variation δa_q in a_q in such a way that the infinite series involving δa remain small. Then the corresponding change δA in A is given by

$$\delta A = \frac{\delta a_q(\theta)\, b_q(\pi - \psi)\, \delta a_p(\pi - \psi)\, b_p(\theta) - A(\theta, \psi)\, \delta a_q(\theta)\, T_{qp}\, \delta a_p(\pi - \psi)}{\{a_q(\theta) + \delta a_q(\theta)\}\, T_{qp} \{a_q(\pi - \psi) + \delta a_p(\pi - \psi)\}}$$

when (40) is used. Hence, the first variation of $A(\theta, \psi)$ as given by (41) is zero when (40) is satisfied. Conversely, it is easy to show that if the first variation of $A(\theta, \psi)$ as determined by (41) vanishes for arbitrary variations of the a_p the equations (40) must be satisfied.

Consequently, if we substitute the appropriate values of b_p, T_{pq} and a_p in (41) we obtain expressions for the distant fields required in (36) and (37) and for β_0 whose first variations are zero for small variations of α_{nm}, γ_{nm} and β_m about their correct values. Variational expressions are capable of giving reasonably accurate results from rough approximations to the unknowns (examples involving integrals will be found in Levine & Schwinger (1948*b*) and Marcuvitz (1951)), but, as has already been stated, no advantage of this has been taken in this paper.

Suppose now that T_{pq} is real and that b_p is real and independent of θ. Then $a_p(\theta)$ is real and independent of θ. Therefore $a_p(\theta) = a_p(\pi-\psi) = a_p$ and

$$\delta A = \frac{(\delta a_q T_{qp} a_p)^2 - (\delta a_q T_{qp} \delta a_p)(a_q T_{qp} a_p)}{(a_q+\delta a_q) T_{qp}(a_p+\delta a_p)}.$$

This form shows that $\delta A \leqslant 0$ if (T) is positive definite and thus that formula (41) always underestimates A in these circumstances.

Consider, in particular, equations (32). When ka is small these equations may be approximated by (see §7)

$$\frac{2\gamma_{nr}}{\dot{j}_{nr}\mathscr{L}^2_{nr}} = \frac{1}{\dot{j}_{nr}} - \sum_{m=1}^{\infty} \frac{\gamma_{nm}}{\dot{j}_{nr}+\dot{j}_{nm}} \quad (r = 1, 2, \ldots), \tag{42}$$

where $\mathscr{L}_{nr} = \lim_{ka\to 0} L_p^{(n)}(\lambda_{nr})$ and the series required in (37) becomes, on omission of the part independent of the summation,

$$\sum_{m=1}^{\infty} \frac{\gamma_{nm}}{\dot{j}_{nm}}. \tag{43}$$

Thus
$$b_p = 1/\dot{j}_{np}, \quad T_{pq} = (\dot{j}_{np}+\dot{j}_{nq})^{-1} + 2\delta_{pq}/\dot{j}_{np}\mathscr{L}^2_{np}, \tag{44}$$

so that T_{pq} is real and b_p is real and independent of θ. Hence a_p is real and independent of θ. Also

$$a_p T_{pq} a_q = 2\sum_{p=1}^{\infty} \frac{a_p^2}{\dot{j}_{np}\mathscr{L}^2_{np}} + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{a_p a_q}{\dot{j}_{np}+\dot{j}_{nq}}. \tag{45}$$

Since $\mathscr{L}_{np} \sim \dot{j}_{np}^{-\frac{1}{2}}$, from (A 14), as $p\to\infty$ the first series in (45) converges and is obviously positive if $a_p = o(p^{-\frac{1}{2}})$ as $p\to\infty$. With regard to the second series we observe that the series $\sum_{p=1}^{\infty} a_p \mathrm{e}^{-\dot{j}_{np}t}$ is absolutely and uniformly convergent for $0 \leqslant t < \infty$ if $a_p = o(p^{-1})$ as $p\to\infty$. Hence the second series is equal to

$$\int_0^\infty \left(\sum_{p=1}^{\infty} a_p \mathrm{e}^{-\dot{j}_{np}t}\right)^2 \mathrm{d}t$$

which is a positive quantity. Hence, for all choices of the a_p such that $a_p = o(p^{-1})$ as $p\to\infty$, (T) is positive definite. In particular this is true of the a_p corresponding to solutions of (42) since the required solutions are those such that $a_p = O(p^{-\frac{7}{6}})$ (see §3). Thus any choice of the a_p, satisfying the above condition, in (41) gives a lower bound to the series (43).

If, now, we choose

$$a_p = c_p \quad (p \leqslant P),$$
$$= 0 \quad (p > P),$$

where

$$b_p = \sum_{p=1}^{P} T_{pq} c_q \quad (p = 1, 2, \ldots, P),$$

b_p and T_{pq} being given by (44), the expression (41) reduces to

$$\sum_{q=1}^{P} c_q b_q = \sum_{q=1}^{P} \frac{c_q}{j_{nq}}.$$

Consequently we can obtain a lower bound to (43) by solving the first P equations of (32) with $\gamma_{nm} = 0$ $(m > P)$ and substituting the values derived thereby in the first P terms of (43).

An application of this theory to the corresponding problem in the theory of the thick plate (Jones 1953 *a*) enables us to improve the estimate made of the series in § 4 of that paper. For we now see that 0·170, the sum of the first four terms, is a definite lower bound for the sum of the whole series. Combining this result with the upper bound obtained in that paper we see that the sum lies between 0·170 and 0·189. Hence the sum of the series (40) in the work referred to is $-0{\cdot}11d$ with a maximum error of 5 %.

7. The distant field when $ka \ll 1$

In this section we shall consider the approximations which can be made when $ka \ll 1$ and the boundary condition is $u = 0$. When $ka \ll 1$,

$$\lambda_{nm} = (j_{nm}/a)\{1 + O(k^2a^2)\} \quad \text{and} \quad L_P^{(n)}(\lambda_{nr}) = \mathscr{L}_{nr}\{1 + o(1)\}.$$

Hence equations (32) become approximately

$$\frac{2\gamma_{nr}}{j_{nr}\mathscr{L}_{nr}^2} = \frac{1}{j_{nr}} - \sum_{m=1}^{\infty} \frac{\gamma_{nm}}{j_{nr} + j_{nm}} \quad (r = 1, 2, \ldots), \tag{46}$$

from which we deduce that $\gamma_{nm} = \text{constant} + o(1)$ (all m).

From the definition (30) we find that

$$\gamma_n = O(k^n a^n) \qquad (n \neq 0)$$
$$= O[\{\ln (ka \sin \theta)\}^{-\frac{1}{2}}] \quad (n = 0)$$

and, from (A 15) and (A 16),

$$L_P^{(n)}(-\mathrm{i}k \cos \psi) = O(1) \qquad (n \neq 0)$$
$$= [-\ln (\tfrac{1}{2}ka \sin \psi)]^{\frac{1}{2}}\{1 + o(1)\} \quad (n = 0).$$

When we recall the definition of $u_n^{(2)}$ we see that if terms of $O(k^2a^2)$ are neglected the only field, apart from the incident field, which needs to be considered at a distance from the origin is that in which $n = 0$. Hence the distant field in $\rho \geqslant a$ is obtained from

$$u^{(0)} - u^{(2)} + u_0(\rho, z) = u^{(0)} - u^{(2)} + \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{R_0(s)\, H_0^{(2)}(\kappa\rho)}{H_0^{(2)}(\kappa a)}\, \mathrm{e}^{sz}\, \mathrm{d}s. \tag{47}$$

Let $R^{(0)}(s)$ be the value of $R_0(s)$ when $\gamma_{0m} = 0$ (all m). Then, for values of s such that $|sa| \ll 1$,

$$\begin{aligned}\frac{R_0(s)\,\mathrm{e}^{sz}}{\gamma_0 L_P^{(0)}(s)} &= \left\{\frac{1}{s+\mathrm{i}k\cos\theta} - \sum_{m=1}^{\infty}\frac{\gamma_{0m}}{s+\lambda_{0m}}\right\}\mathrm{e}^{sz} \\ &= \{(s+\mathrm{i}k\cos\theta)^{-1} - \zeta\}\,\mathrm{e}^{sz} \\ &= \{1-(s+\mathrm{i}k\cos\theta)\,\zeta\}\,\mathrm{e}^{sz}\,(s+\mathrm{i}k\cos\theta)^{-1} \\ &= \frac{R^{(0)}(s)}{\gamma_0 L_P^{(0)}(s)}\,\mathrm{e}^{sz-(s+\mathrm{i}k\cos\theta)\zeta},\end{aligned} \tag{48}$$

where $\zeta = \sum_{m=1}^{\infty}\gamma_{0m}/\lambda_{0m}$.

Now, in determining the distant field from the integral of (47) the only part of the contour which is significant is that near the saddle-point and, in that neighbourhood, $|sa| = O(ka)$. Hence we may substitute for $R_0(s)$ in (47) from (48). But the integrand is now, apart from the term involving ζ, just that which occurs in the diffraction by a semi-infinite hollow cylinder. The effect of the ζ term is to make the semi-infinite hollow cylinder occupy the position $\rho = a$, $-\infty \leqslant z \leqslant \zeta$.

Thus the field at a point some distance from the end of the rod is the same as that at the point when a semi-infinite hollow cylinder of the same diameter but longer by an amount $\zeta = \sum_{m=1}^{\infty}\frac{\gamma_{0m}}{\lambda_{0m}}$ is subject to the same incident field.

To estimate ζ we have solved the first two of equations (46) under the assumption that $\gamma_{0m} = 0$ $(m>2)$. We find $\gamma_{01} = 0{\cdot}178$, $\gamma_{02} = 0{\cdot}075$ and hence $\gamma_{01}/\lambda_{01}+\gamma_{02}/\lambda_{02} = 0{\cdot}0872a$. Therefore our approximation to ζ is $0{\cdot}087a$. It follows from the theory at the end of the preceding section that this approximate value is below the true value of ζ. A like approximation in the theory of the thick plate is about 10 % low, and if we assume that the same is true here it would appear that ζ is very nearly $0{\cdot}1a$.

It follows that, to the degree of approximation adopted above, the rod behaves, as far as the distant field is concerned, as a hollow cylinder longer by an amount $0{\cdot}1a$. This result, being independent of θ, holds for any incident field which can be constructed from a spectrum of plane waves.

In determining γ_{01} and γ_{02} it is necessary to compute $\mathscr{L}_{0r}$, i.e.

$$\exp\frac{j_{0r}}{\pi}\int_0^\infty \frac{\ln\{2I_0(t)\,K_0(t)\}}{t^2+j_{0r}^2}\,\mathrm{d}t$$

from (A 13).

Part of the computation was carried out by Mr D. F. Ferguson. The results are

$$\mathscr{L}_{01} = 0{\cdot}6318 \quad \text{and} \quad \mathscr{L}_{02} = 0{\cdot}423.$$

A similar result when the boundary condition is $\partial u/\partial \nu = 0$ does not hold. It can be shown that for any given n the distant field behaves as that due to a slightly longer hollow cylinder (together with a doublet at the origin when $n = 0$), but there seems to be no reason why the additional length should be independent of n. Indeed, if we reject terms of $O(k^4a^4)$ it is necessary to retain the field of both $n = 0$ and $n = 1$, since both $u_0^{(1)}$ and $u_1^{(1)}$ are $O(k^2a^2)$, and so the rod behaves as a slightly longer tube only if the lengths associated with $n = 0$, $n = 1$ are the same. Now, taking as a first approximation only α_{00}, α_{01} and α_{11} as non-zero, we find $\alpha_{00} = (1+\cos\theta)^{-1}$ so that α_{01} is dependent on θ, whereas α_{11} is not. Hence the additional lengths for $n = 0$, $n = 1$ will be different in general and the rod will not behave as a slightly longer tube.

8. The pressure on the rod due to a pressure pulse

The analysis in the preceding sections has dealt entirely with the case of a harmonic incident plane wave, but from it can be deduced results concerning the effects when a pulse is incident on a rigid rod. We shall consider only sound pulses and restrict the analysis to the case when the pulse meets the end of the rod head-on, i.e. the angle of incidence is π.

Let p_0, the pressure in the incident sound pulse, be given by

$$p_0 = H(t+z/a_0),$$

where a_0 is the speed of sound, t is the time and $H(x)$ is the Heaviside unit function. On taking a Laplace transform with respect to t we obtain

$$\int_{-\infty}^{\infty} p_0\,\mathrm{e}^{-qt}\,\mathrm{d}t = (1/q)\,\mathrm{e}^{qz/a_0} \quad (\mathscr{R}(q)>0).$$

If now we put $q/a_0 = ik$ the incident wave is the same as that of § 2, apart from the factor $1/q$ and the transform of the total pressure satisfies (1) ($\mathscr{I}(k)<0$ since $\mathscr{R}(q)>0$). The normal derivative of the transform of the total pressure vanishes on the surface of the rod since the normal derivative of the total pressure is zero there. The problem is consequently the same as that of § 2. It follows from § 5 that the transform of the average pressure on the end of the rod is

$$2\{1-B_0(q)\}/q,$$

where $B_0(q)$ is the value of β_0 when expressed in terms of q. Hence the average pressure on the end of the rod at time t (>0) is

$$\frac{1}{\pi \mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty}\{1-B_0(q)\}\,\mathrm{e}^{qt}\,\mathrm{d}q/q \quad (d>0).$$

The integrand has a simple pole at $q = 0$ with residue $\frac{1}{2}$ since $\beta_0 = \frac{1}{2}$ when $ka = 0$. Thus the above integral may be written as

$$1+\frac{1}{\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\{\tfrac{1}{2}-B_0(q)\}\,\mathrm{e}^{qt}\,\mathrm{d}q/q.$$

Put $q = \mathrm{i}ka_0 = \mathrm{i}\alpha a_0/a$, where $\alpha = ka$ and $T = a_0t/a$, so that $T = 1$ corresponds to the time taken by a sound wave to travel a distance equal to the radius of the rod. Then the average pressure on the end of the rod is

$$1+\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{1-2\beta_0}{\alpha}\,\mathrm{e}^{\mathrm{i}\alpha T}\,\mathrm{d}\alpha,$$

where β_0 is the same function of α as that obtained by solving equations (25).

Now β_0 has already been determined approximately in § 5 when $\alpha>0$. Let the value of β_0 so obtained be given by

$$2(1-\beta_0) = F(\alpha)+\mathrm{i}G(\alpha).$$

Then, when $\alpha<0$,

$$2(1-\beta_0) = F(-\alpha)-\mathrm{i}G(-\alpha),$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and hence the average pressure on the end of the rod is

$$1+\frac{1}{\pi}\int_0^\infty \frac{1}{\alpha}\left[\{F(\alpha)-1\}\sin\alpha T+G(\alpha)\cos\alpha T\right]\mathrm{d}\alpha. \tag{49}$$

Table 1. The average pressure at various times

T		T	
0	1·974	2	0·915
$\frac{1}{2}$	1·616	3	0·991
1	1·294	4	1·005
$\frac{3}{2}$	1·038		

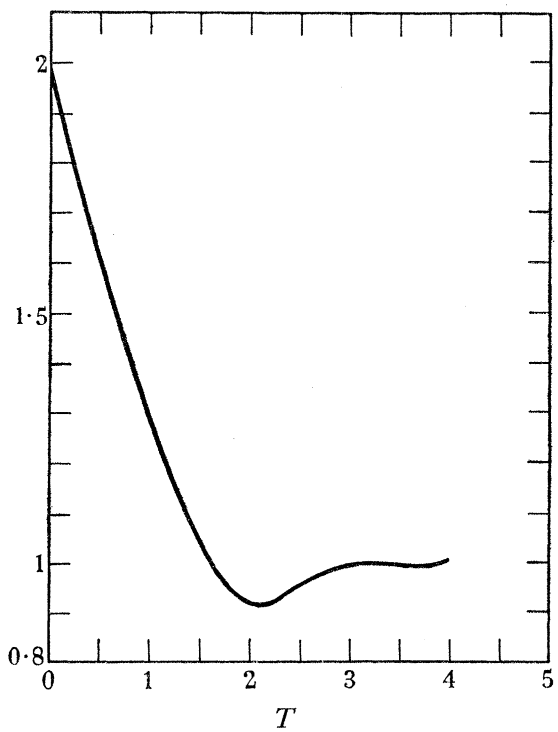

Figure 4. The average pressure on the end of the rod with an incident pressure pulse.

The way that $F(\alpha)$ behaves for $\alpha\leqslant 10$ can be seen from figure 2 since $c_0 = F(\alpha)+1$; the behaviour of $G(\alpha)$ may be deduced from figures 1 and 2 since figure 1 shows $(F^2+G^2)^{\frac{1}{2}}$. The values of F and G have not been computed for $\alpha>10$, but it is clear from the figures that, in this range, we can take $F(\alpha) = 2$ and $G(\alpha) = 0$. This approximation should not produce an error of more than 5%. On introducing this approximation the average pressure on the end of the rod becomes

$$\frac{3}{2}-\frac{2}{\pi}\mathrm{Si}\,(10T)+\frac{1}{\pi}\int_0^{10}\frac{1}{\alpha}\{F(\alpha)\sin\alpha T+G(\alpha)\cos\alpha T\}\mathrm{d}\alpha, \tag{50}$$

where $\mathrm{Si}\,(x) = \frac{1}{2}\pi-\int_x^\infty \frac{\sin x}{x}\mathrm{d}x$. This formula gives the correct value of 1 as $T\to\infty$.

The integral was calculated by replacing F and G by parabolic approximations over the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, ... The results thus obtained are shown in table 1 and figure 4.

The pressure drops rapidly to its final value of 1, differing by less than 10 % from this value for $T>2$. It can be seen that the approximation described above is justified. The correct value at $T=0$ should be 2 (which differs from the approximate value by 3 %), since just after the incident wave hits the end of the rod the pressure in the neighbourhood of the end is the same as when a pulse is reflected at a plane wall. The waves diffracted from the edges then spread across the end reducing the pressure rapidly, and it appears that by the time ($T=2$) these waves produce diffracted waves the average pressure is not greatly different from its final value.

It may be noted that the average pressure in $0 \leqslant T \leqslant 2$, as given by table 1, is equal to

$$\{0{\cdot}915+0{\cdot}745(2-T)^2\}^{\frac{1}{2}}$$

correct to within 1 %.

The computation necessary for table 1 was carried out by Mr D. F. Ferguson.

The foregoing theory applies to the semi-infinite rod, but the values obtained for the pressure will continue to hold for a rod of finite length until the waves diffracted at the second end reach the first end. Thus, if the rod is of length l, the average pressure given by (49) and (50) will be correct for $0 \leqslant T \leqslant 2l/a$.

It is clear also that we may insert a rigid plane along $\phi=0$ without affecting our solution; we thus obtain the pressure on an obstacle of semicircular cross-section placed on the ground due to a pressure pulse travelling along the ground.

9. Approximate evaluation of the 'split' functions

The quantities $K_P^{(0)}(\kappa_{0m})$ $(m=0,1,2)$ have been computed on the Manchester Digital Computer and the results are shown in table 2. The analysis, including a method for computing infinite integrals, necessary to convert $K_P^{(0)}$ into a form suitable for the computer, was carried out by Mr A. M. Turing. The subsequent routine was programmed for the Digital Computer by Mr R. A. Brooker. Details of their work appear in a separate paper (Brooker & Turing, unpublished).

Some values of $K_P^{(0)}(\kappa_{00})$ have been given in graphical form by Levine & Schwinger (1948*a*). The values we obtain agree, apart from one exception, with those of Levine & Schwinger to the accuracy with which the graph can be read. The exception is $I_{0,0}$ ($I_{0,0}$ is defined by (51) and (53) below); we obtain 0·6128i, whereas Levine & Schwinger give 0·6133i.

Various approximate results were required in the preceding analysis and also provided a useful check on the computation; these are obtained in the following work. When $k_i=0$ we have from (A 1)

$$\ln K_P^{(0)}(w) = -\frac{1}{2\pi \mathrm{i}}\mathscr{P}\int_{\Gamma_0}\frac{\ln\{-\pi \mathrm{i} J_0'(\kappa a)\,H_0^{(2)\prime}(\kappa a)\}}{s-w}\,\mathrm{d}s,$$

where w lies to the right of the contour Γ_0 which goes from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ to the right of $-\mathrm{i}k$ and to the left of $\mathrm{i}k$.

The zeros of $J_0'(ka)$ in the negative half-plane lie to the left of Γ_0 and those of the positive half-plane to the right. After rotating the complex plane through a right angle and replacing ka by α we obtain

$$\ln K_P^{(0)}(w) = -\frac{1}{2\pi \mathrm{i}}\mathscr{P}\int_{\Gamma}\frac{\ln[-\pi \mathrm{i} J_1\{\surd(\alpha^2-t^2)\}\,H_1^{(2)}\{\surd(\alpha^2-t^2)\}]}{t-aw\,\mathrm{e}^{-\frac{1}{2}\pi \mathrm{i}}}\,\mathrm{d}t,$$

TABLE 2

	$K_P^{(0)}(ik)$		$K_P^{(0)}(\kappa_{01})$		$K_P^{(0)}(\kappa_{02})$	
ka	$\mathscr{R}$	$\mathscr{I}$	$\mathscr{R}$	$\mathscr{I}$	$\mathscr{R}$	$\mathscr{I}$
0	1	0	0·4533	−0	0·3594	−0
0·25	0·9747	−0·1501	·4556	−0·0003	·3604	−0·0001
0·5	·9078	·2747	·4618	·0021	·3631	·0009
0·75	·8180	·3654	·4715	·0066	·3671	·0028
1	0·7206	−0·4235	0·4836	−0·0147	0·3721	−0·0061
1·25	·6252	·4541	·4976	·0272	·3778	·0110
1·5	·5375	·4634	·5124	·0449	·3837	·0177
1·75	·4601	·4568	·5270	·0687	·3894	·0264
2	0·3942	−0·4391	0·5395	−0·0996	0·3946	−0·0371
2·25	·3394	·4139	·5476	·1385	·3986	·0500
2·5	·2952	·3839	·5480	·1858	·4008	·0649
2·75	·2606	·3510	·5356	·2413	·4001	·0819
3	0·2348	−0·3163	0·5028	−0·3026	0·3954	−0·1004
3·25	·2173	·2799	·4378	·3624	·3850	·1196
3·5	·2086	·2405	·3206	·3982	·3657	·1376
3·75	·2145	·1888	·1101	·3080	·3275	·1483
4	0·2867	−0·1903	0·3440	−0·1148	0·3145	−0·0976
4·25	·2900	·2208	·3761	·2052	·3331	·0884
4·5	·2790	·2391	·3553	·2531	·3488	·0902
4·75	·2625	·2493	·3223	·2768	·3628	·0989
5	0·2439	−0·2532	0·2880	−0·2851	0·3748	−0·1137
5·25	·2253	·2518	·2562	·2831	·3837	·1344
5·5	·2078	·2464	·2286	·2742	·3878	·1608
5·75	·1922	·2376	·2055	·2606	·3846	·1929
6	0·1789	−0·2260	0·1870	−0·2437	0·3705	−0·2294
6·25	·1684	·2119	·1733	·2244	·3399	·2676
6·5	·1612	·1954	·1644	·2026	·2842	·2997
6·75	·1588	·1750	·1616	·1770	·1886	·3030
7	0·1705	−0·1388	0·1749	−0·1328	0·0167	−0·1197
7·25	·2068	·1640	·2202	·1614	·2805	·1131
7·5	·2054	·1801	·2194	·1809	·2921	·1750
7·75	·1982	·1901	·2111	·1930	·2750	·2088
8	0·1884	−0·1957	0·1997	−0·1998	0·2507	−0·2256
8·25	·1776	·1978	·1872	·2024	·2258	·2313
8·5	·1668	·1970	·1748	·2015	·2029	·2295
8·75	·1565	·1937	·1631	·1977	·1831	·2225
9	0·1473	−0·1882	0·1527	−0·1915	0·1666	−0·2119
9·25	·1395	·1809	·1439	·1832	·1536	·1987
9·5	·1334	·1717	·1372	·1730	·1443	·1832
9·75	·1298	·1604	·1332	·1604	·1391	·1654
10	0·1305	−0·1452	0·1338	−0·1437	0·1398	−0·1426

where Γ is the contour in figure 5 and $\surd(\alpha^2-t^2) = e^{\frac{1}{2}\pi i}\surd(t^2-\alpha^2)$, $\surd(t^2-\alpha^2)$ being equal to $\alpha\, e^{-\frac{1}{2}\pi i}$ when $t = 0$.

FIGURE 5

We may also obtain a similar result by rotating the s-plane through a right angle in the opposite direction. An addition of the two formulae gives

$$\ln K_P^{(0)}(w) = \frac{aw}{2\pi}\int_\Gamma \frac{\ln[-\pi i J_1\{\surd(\alpha^2-t^2)\}H_1^{(2)}\{\surd(\alpha^2-t^2)\}]}{t^2+a^2w^2}\,dt,$$

the points iaw and $-iaw$ lying above and below Γ respectively.

We write
$$\ln K_P^{(0)}(\kappa_{0\nu}) = \mathrm{i}a\kappa_{0\nu} I_{\alpha,\nu}, \tag{51}$$
where
$$I_{\alpha,\nu} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\ln[-\pi\mathrm{i}\, J_1\{\surd(\alpha^2-t^2)\}\, H_1^{(2)}\{\surd(\alpha^2-t^2)\}]}{t^2+j_{0\nu}'^2-\alpha^2}\,\mathrm{d}t. \tag{52}$$

The zeros of J_1 which lie above Γ are given by $t = \mathrm{i}\surd(j_{0\mu}'^2-\alpha^2)$, where
$$\begin{aligned}\surd(j_{0\mu}'^2-\alpha^2) &= \surd(j_{0\mu}'^2-\alpha^2) \quad \text{if} \quad j_{0\mu}'^2>\alpha^2,\\ &= \mathrm{i}\surd(\alpha^2-j_{0\mu}'^2) \quad \text{if} \quad \alpha^2>j_{0\mu}'^2.\end{aligned}$$
(It will be assumed that there is no μ such that $j_{0\mu}' = \alpha$.) The branch lines from these points (except $-\alpha$) are drawn to make a small angle with the negative real axis; the branch line from $-\alpha$ is taken along the negative real axis. The contour is then deformed into the upper half-plane over the branch lines. The contributions from the branch lines (except that from $t=-\alpha$) are easily evaluated and we obtain
$$I_{\alpha,\nu} = \lim_{m\to\infty}\left[\frac{1}{2}+\sum_{\mu=1}^{m} R_{\alpha,\nu}(j_{0\mu}') + \frac{1}{2\pi\mathrm{i}}\int_0^R \frac{\ln[K_1(u)/\{K_1(u)+\mathrm{i}\pi I_1(u)\}]\,u\,\mathrm{d}u}{(u^2+j_{0\nu}'^2)\,(\alpha^2+u^2)^{\frac{1}{2}}}\right],$$
where
$$j_{0m}' < R < j_{0,m+1}', \quad J_\nu(z\,\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}) = \mathrm{e}^{-\frac{1}{2}\nu\pi\mathrm{i}} I_\nu(z) \quad \text{and} \quad K_\nu(z) = -\tfrac{1}{2}\pi\mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\nu\pi\mathrm{i}} H_\nu^{(2)}(z\,\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}).$$
The quantity $R_{\alpha,\nu}$ is defined by
$$\begin{aligned} 2F_3(j_{0\nu}')\,R_{\alpha,\nu}(j_{0\mu}') &= -\mathrm{i}F_1(j_{0\mu}', j_{0\nu}') && (j_{0\mu}'>j_{0\nu}'\geqslant\alpha),\\ &= -\pi-\mathrm{i}F_1(j_{0\nu}', j_{0\mu}') && (j_{0\nu}'>j_{0\mu}'\geqslant\alpha),\\ &= -\pi\mathrm{i}+2\mathrm{i}\tan^{-1}\{F_3(j_{0\mu}')/F_3(j_{0\nu}')\} && (j_{0\mu}'>\alpha>j_{0\nu}'),\\ &= -2\tan^{-1}\{F_3(j_{0\nu}')/F_3(j_{0\mu}')\} && (j_{0\nu}'>\alpha>j_{0\mu}'),\\ &= -F_2(j_{0\mu}', j_{0\nu}') && (\alpha>j_{0\nu}'>j_{0\mu}'),\\ &= -\pi\mathrm{i}-F_2(j_{0\nu}', j_{0\mu}') && (\alpha>j_{0\mu}'>j_{0\nu}'),\\ &= -\pi-\mathrm{i}\ln\{4(j_{0\nu}'^2-\alpha^2)/j_{0\nu}'^2\} && (j_{0\mu}'=j_{0\nu}'>\alpha),\\ &= -\ln\{4(\alpha^2-j_{0\nu}'^2)/j_{0\nu}'^2\} && (j_{0\mu}'=j_{0\nu}'<\alpha),\end{aligned}$$
where
$$F_1(a,b) = \ln[\{\surd(a^2-\alpha^2)+\surd(b^2-\alpha^2)\}^2/(a^2-b^2)],$$
$$F_2(a,b) = \ln[\{\surd(\alpha^2-a^2)+\surd(\alpha^2-b^2)\}^2/(b^2-a^2)],$$
and
$$F_3(a) = \surd|a^2-\alpha^2|.$$
Let
$$I_{0,\nu} = \lim_{\alpha\to 0} I_{\alpha,\nu} \tag{53}$$
and $R_{0,\nu} = \lim_{\alpha\to 0} R_{\alpha,\nu}$; then
$$I_{\alpha,\nu} - I_{0,\nu} = \lim_{m\to\infty}\sum_{\mu=1}^{m}\{R_{\alpha,\nu}(j_{0\mu}') - R_{0,\nu}(j_{0\mu}')\} + \frac{1}{2\pi\mathrm{i}}\int_0^\infty \left\{\frac{u}{(\alpha^2+u^2)^{\frac{1}{2}}}-1\right\}\frac{\ln[K_1(u)/\{K_1(u)+\mathrm{i}\pi I_1(u)\}]}{u^2+j_{0\nu}'^2}\,\mathrm{d}u.$$
An integration by parts converts this into a formula derived by Turing a different way and which is fundamental to his method of calculating $I_{\alpha,\nu}$. This formula is
$$I_{\alpha,\nu} - I_{0,\nu} = \lim_{m\to\infty}\sum_{\mu=1}^{m}\{R_{\alpha,\nu}(j_{0\mu}') - R_{0,\nu}(j_{0\mu}')\} + \frac{1}{2}\int_0^\infty \frac{S_{\alpha,\nu}(u) - S_{0,\nu}(u)}{uK_1(u)\,\{K_1(u)+\mathrm{i}\pi I_1(u)\}}\,\mathrm{d}u,$$

where $$2F_3(j'_{0\nu})\,S_{\alpha,\nu}(u) = -F_2(\mathrm{i}u, j'_{0\nu}) \qquad (\alpha \geqslant j'_{0\nu}),$$
$$= -2\tan^{-1}\{F_3(j'_{0\nu})/F_3(\mathrm{i}u)\} \qquad (j'_{0\nu} \geqslant \alpha),$$

and $$S_{0,\nu}(u) = (1/j'_{0\nu})\{\tan^{-1}(u/j'_{0\nu}) - \tfrac{1}{2}\pi\}.$$

Consider now the integral

$$\int_{-\infty}^{\infty} \frac{u}{u^2+\beta^2} \ln\left\{\left(\frac{2u}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{u} K_1(u)\right\} \mathrm{d}u,$$

where β is real and positive and the line of integration lies below the branch lines. The contour may be deformed to infinity in the lower half-plane, the only contribution being from the pole $u = -\mathrm{i}\beta$. Hence the integral is equal to

$$-\pi\mathrm{i}\ln\left\{\left(\frac{2\beta}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi - \mathrm{i}\beta} K_1(\beta\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi})\right\}.$$

But the integral may also be written, by changing the sign of the variable in the range $(-\infty, 0)$, as

$$\int_0^{\infty} \frac{u}{u^2+\beta^2} \ln \frac{\mathrm{e}^{2u} K_1(u)}{\mathrm{i}K_1(u)+\pi I_1(u)} \mathrm{d}u.$$

Hence $$\int_0^{\infty} \frac{u}{u^2+\beta^2} \ln \frac{\mathrm{i}K_1(u)+\pi I_1(u)}{\mathrm{e}^{2u} K_1(u)} \mathrm{d}u = \pi\mathrm{i}\ln\left\{\left(\frac{\pi\beta}{2}\right)^{\frac{1}{2}} \mathrm{e}^{(\frac{3}{4}\pi-\beta)\mathrm{i}} H_1^{(1)}(\beta)\right\}.$$

This formula was obtained by a different process by Levine & Schwinger (1948*a*, equation (V.37)).

Therefore, on taking the imaginary part of both sides, we have

$$\int_0^{\infty} \frac{u}{u^2+\beta^2} \tan^{-1} \frac{K_1(u)}{\pi I_1(u)} \mathrm{d}u = \tfrac{1}{2}\pi \ln\left[\tfrac{1}{2}\pi\beta\{J_1^2(\beta)+Y_1^2(\beta)\}\right]. \tag{54}$$

In particular it follows, by allowing $\beta \to \infty$, that

$$\int_0^{\infty} u \tan^{-1} \frac{K_1(u)}{\pi I_1(u)} \mathrm{d}u = \frac{3\pi}{16}, \tag{55}$$

and, by allowing $\beta \to 0$, that

$$\int_{\epsilon}^{\infty} \frac{1}{u} \tan^{-1} \frac{K_1(u)}{\pi I_1(u)} \mathrm{d}u = \tfrac{1}{2}\pi \ln(2/\pi\epsilon), \tag{56}$$

where ϵ is a small positive number.

Now
$$I'_{\alpha,\nu} = \mathscr{R} I_{\alpha,\nu}$$
$$= \tfrac{1}{2} + \mathscr{R}\sum_{\mu=1}^{M-1} R_{\alpha,\nu}(j'_{0\mu}) + \lim_{\epsilon\to 0}\left[\tfrac{1}{4}S_{\alpha,\nu}(\epsilon) + \frac{1}{2\pi}\int_{\epsilon}^{\infty} \frac{u\tan^{-1}\{K_1(u)/\pi I_1(u)\}}{(u^2+j'^2_{0\nu})(\alpha^2+u^2)^{\frac{1}{2}}} \mathrm{d}u\right],$$

where M is the smallest integer such that $j'_{0M} > \alpha$ and $j'_{0M} > j'_{0\nu}$. If α is large we may replace $(\alpha^2+u^2)^{-\frac{1}{2}}$ by $1/\alpha - \frac{1}{2}u^2/\alpha^3$ when u takes moderate values. When u is large

$$\tan^{-1}\{K_1(u)/\pi I_1(u)\} = O(\mathrm{e}^{-2u}),$$

so that the error in taking the same form for $(\alpha^2+u^2)^{-\frac{1}{2}}$ when u is large will produce a negligible error in the integral in $I'_{\alpha,\nu}$. Hence, for large α,

$$I'_{\alpha,\nu} = \tfrac{1}{2}+\mathscr{R}\sum_{\mu=1}^{M-1} R_{\alpha,0}(j'_{0\mu}) - \frac{1}{4\alpha}\ln \pi\alpha - \frac{3}{64\alpha^3} \quad (\nu=0) \tag{57}$$

$$= \tfrac{1}{2}+\mathscr{R}\sum_{\mu=1}^{M-1} R_{\alpha,\nu}(j'_{0\mu}) + \tfrac{1}{4}S_{\alpha,\nu}(0) - \frac{3}{64\alpha^3} + \frac{1}{4\alpha}\left(1+\frac{1}{2}\frac{j'^2_{0\nu}}{\alpha^2}\right)\ln\{\tfrac{1}{2}\pi j'_{0\nu} Y_1^2(j'_{0\nu})\} \quad (\nu\neq 0), \tag{58}$$

when use is made of (54), (55) and (56).

It was found that the values of $I'_{\alpha,\nu}$ obtained from (57) and (58) differed from the computed value by less than 1 % when $\alpha \geqslant 2$.

The above approximation is of course valueless in the range $\alpha<1$. For this range an approximation may be derived by starting from (52) and observing that

$$I'_{\alpha,\nu} = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \tan^{-1}\frac{J_1\{\surd(\alpha^2-t^2)\}}{Y_1\{\surd(\alpha^2-t^2)\}}\frac{\mathrm{d}t}{t^2+j'^2_{0\nu}-\alpha^2} \quad \text{when} \quad \alpha<j'_{01}$$

$$= \frac{1}{\pi}\int_0^{\alpha}\frac{u\tan^{-1}\{J_1(u)/Y_1(u)\}}{(j'^2_{0\nu}-u^2)\,(\alpha^2-u^2)^{\frac{1}{2}}}\,\mathrm{d}u.$$

On using the expansions for Bessel functions of small argument we find that, for small α,

$$I'_{\alpha,\nu} \approx -\alpha^3/6j'^2_{0\nu} \quad (\nu\neq 0) \tag{59}$$

$$\approx \tfrac{1}{4}\{\alpha+\tfrac{1}{3}\alpha^3\ln\alpha+\tfrac{1}{3}(\gamma-\tfrac{19}{12})\,\alpha^3\} \quad (\nu=0), \tag{60}$$

where $\gamma = 0{\cdot}5772\ldots$ is Euler's constant.

The author gratefully acknowledges the considerable debt he owes the late Mr A. M. Turing, F.R.S., and Mr R. A. Brooker for their work in the computation involved in this paper; he is also indebted to Mr D. F. Ferguson for additional computation.

Appendix

By applying Cauchy's theorem to the contour C in figure 6 it is easy to show that, since $J'_n(\kappa a)\,H_n^{(2)\prime}(\kappa a)$ has no zeros in $-k_i<\sigma<k_i$ (the proof that $H_n^{(2)\prime}(z)$ has no zeros for $0\geqslant \arg z\geqslant -\pi$ is similar to that given by Watson (1944) for $H_n^{(2)}(z)$),

$$\ln K_P^{(n)}(w) = -\frac{1}{2\pi\mathrm{i}}\mathscr{P}\int_{\sigma_1-\mathrm{i}\infty}^{\sigma_1+\mathrm{i}\infty}\frac{\ln\{-\pi\mathrm{i}J'_n(\kappa a)\,H_n^{(2)\prime}(\kappa a)\}}{s-w}\,\mathrm{d}s, \tag{A 1}$$

$$\ln K_N^{(n)}(w) = -\frac{1}{2\pi\mathrm{i}}\mathscr{P}\int_{\sigma_2-\mathrm{i}\infty}^{\sigma_2+\mathrm{i}\infty}\frac{\ln\{-\pi\mathrm{i}J'_n(\kappa a)\,H_n^{(2)\prime}(\kappa a)\}}{s-w}\,\mathrm{d}s, \tag{A 2}$$

where in the first integral $\mathscr{R}(w)>\sigma_1$, $-k_i<\sigma_1<k_i$ and, in the second, $\mathscr{R}(w)<\sigma_2$, $-k_i<\sigma_2<k_i$ and the symbol $\mathscr{P}$ indicates that the integral is to be calculated as the limit when both ends of the contour tend to the infinity at the same time (see Pearson 1953). It follows that

$$\ln K_P^{(n)}(w)+\ln K_N^{(n)}(-w) = 0,$$

and hence that

$$K_P^{(n)}(w)\,K_N^{(n)}(-w) = 1. \tag{A 3}$$

By a process similar to that described at the beginning of § 8 we obtain

$$\ln K_P^{(n)}(w) = \frac{aw}{2\pi}\int_\Gamma \frac{\ln\left[-\pi \mathrm{i} J_n'\{\surd(\alpha^2-t^2)\}\, H_n^{(2)\prime}\{\surd(\alpha^2-t^2)\}\right]}{t^2+a^2w^2}\,\mathrm{d}t,$$

where $\alpha = ka$.

When $|wa| \gg 1$ we have

$$\ln K_P^{(n)}(w) = \frac{aw}{\pi}\int_N^\infty \frac{\ln\left[-2I_n'\{\surd(t^2-\alpha^2)\}\, K_n'\{\surd(t^2-\alpha^2)\}\right]}{t^2+a^2w^2}\,\mathrm{d}t + O(1/|wa|),$$

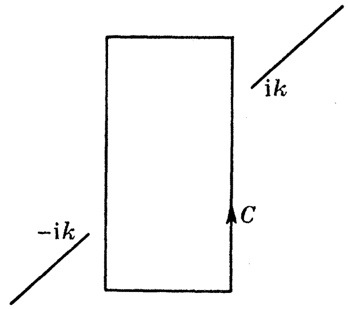

FIGURE 6

where N is a fixed large number. In the integral we can use the asymptotic formulae for the Bessel functions and hence

$$\ln K_P^{(n)}(w) \sim \frac{aw}{\pi}\int_N^\infty \frac{-\ln t}{t^2+a^2w^2}\,\mathrm{d}t + O\left(\frac{1}{|wa|}\ln|wa|\right)$$

$$\sim -\tfrac{1}{2}\ln(aw) + O\left(\frac{1}{|wa|}\ln|wa|\right).$$

Therefore
$$K_P^{(n)}(w) \sim (aw)^{-\frac{1}{2}}\left\{1+O\left(\frac{1}{|wa|}\ln|wa|\right)\right\} \tag{A 4}$$

as $|wa| \to \infty$ in the positive half-plane.

When $ka \to 0$ and $aw \to W$ $(\neq 0)$ it is clear that

$$\ln K_P^{(n)}(w) \to \frac{W}{\pi}\int_0^\infty \frac{\ln\{-2I_n'(t)\,K_n'(t)\}}{t^2+W^2}\,\mathrm{d}t \tag{A 5}$$

and
$$K_P^{(n)}(w) = O(1). \tag{A 6}$$

If $w = -\mathrm{i}k\cos\psi$ this formula does not hold; in this case

$$\ln K_P^{(n)}(w) = \frac{-\mathrm{i}\alpha\cos\psi}{2\pi}\int_{-\epsilon}^{\epsilon} \frac{\ln\left[-\pi\mathrm{i}J_n'\{\surd(\alpha^2-t^2)\}\, H_n^{(2)\prime}\{\surd(\alpha^2-t^2)\}\right]}{t^2-\alpha^2\cos^2\psi}\,\mathrm{d}t + O(\alpha), \tag{A 7}$$

where ϵ is a fixed small quantity greater than α and the contour passes above α, $-\alpha\cos\psi$ and below $-\alpha$, $\alpha\cos\psi$. When $n \neq 0$ this may be approximated by

$$-\frac{\mathrm{i}\alpha\cos\psi}{2\pi}\int_{-\epsilon}^{\epsilon}\frac{\ln\{-n(n+1)/(\alpha^2-t^2)\}}{t^2-\alpha^2\cos^2\psi}\,\mathrm{d}t$$

$$= \tfrac{1}{2}\ln\{-n(n+1)/\alpha^2\} + O(1).$$

Hence, as $\alpha \to 0$,
$$K_P^{(n)}(-\mathrm{i}k\cos\psi) \approx \text{constant}/\alpha \quad (n \neq 0). \tag{A 8}$$

When $n=0$ the right-hand side of (A 7) is $O(\alpha)$ and so

$$K_P^{(0)}(-\mathrm{i}k\cos\psi)\approx 1. \tag{A 9}$$

The split functions corresponding to $L(s)$ may be dealt with in a similar way. We have

$$\ln L_P^{(n)}(w) = \frac{1}{2\pi\mathrm{i}}\mathscr{P}\int_{\sigma_1-\mathrm{i}\infty}^{\sigma_1+\mathrm{i}\infty}\frac{\ln\{-\pi\mathrm{i}J_n(\kappa a)\,H_n^{(2)}(\kappa a)\}}{s-w}\,\mathrm{d}s \tag{A 10}$$

and

$$L_P^{(n)}(w)\,L_N^{(n)}(-w) = 1. \tag{A 11}$$

Note that $L_P^{(1)}(w)=K_P^{(0)}(w)$.

The asymptotic behaviour is given by

$$L_P^{(n)}(w)\sim(aw)^{-\frac{1}{2}}\left\{1+O\left(\frac{1}{|wa|}\ln|wa|\right)\right\} \tag{A 12}$$

as $|wa|\to\infty$ in the positive half-plane. Also when $ka\to 0$ and $wa\to W(\neq 0)$

$$\ln L_P^{(n)}(w)\to\frac{W}{\pi}\int_0^\infty\frac{\ln\{2I_n(t)\,K_n(t)\}}{t^2+W^2}\,\mathrm{d}t \tag{A 13}$$

and

$$L_P^{(n)}(w) = O(1). \tag{A 14}$$

If $w=-\mathrm{i}k\cos\psi$ we obtain

$$L_P^{(n)}(-\mathrm{i}k\cos\psi)\approx O(1) \qquad (n\neq 0) \tag{A 15}$$

$$\approx\{-\ln(\tfrac{1}{2}ka\sin\psi)\}^{\frac{1}{2}} \qquad (n=0) \tag{A 16}$$

as $ka\to 0$.

References

Clemmow, P. C. 1950 *Quart. J. Mech.* **3**, 241.
Jones, D. S. 1952*a* *Proc. Camb. Phil. Soc.* **48**, 118.
Jones, D. S. 1952*b* *Quart. J. Mech.* **3**, 363.
Jones, D. S. 1952*c* *Proc. Camb. Phil. Soc.* **48**, 733.
Jones, D. S. 1953*a* *Proc. Roy. Soc.* A, **217**, 153.
Jones, D. S. 1953*b* *Proc. Camb. Phil. Soc.* **49**, 668.
Levine, H. & Schwinger, J. 1948*a* *Phys. Rev.* **73**, 383.
Levine, H. & Schwinger, J. 1948*b* *Phys. Rev.* **74**, 958.
Lommel, E. C. J. von 1871 *Math. Ann.* **4**, 103.
Lommel, E. C. J. von 1879 *Math. Ann.* **14**, 520.
Marcuvitz, N. 1951 *Waveguide handbook.* New York: McGraw-Hill.
Meixner, J. 1949 *Ann. Phys., Lpz.* (6), **6**, 1.
Pearson, J. D. 1953 *Proc. Camb. Phil. Soc.* **49**, 659.
Watson, G. N. 1944 *Theory of Bessel functions*, 2nd ed. Cambridge University Press.
Williams, W. E. 1954*a* *Proc. Camb. Phil. Soc.* **50**, 309.
Williams, W. E. 1954*b* Thesis, University of Manchester.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES